

MULTIPLIER ALGEBRAS OF COMPLETE NEVANLINNA-PICK SPACES: DILATIONS, BOUNDARY REPRESENTATIONS AND HYPERRIGIDITY

RAPHAËL CLOUÂTRE AND MICHAEL HARTZ

ABSTRACT. We study reproducing kernel Hilbert spaces on the unit ball with the complete Nevanlinna-Pick property through the representation theory of their algebras of multipliers. We give a complete description of the representations in terms of the reproducing kernels. While representations always dilate to $*$ -representations of the ambient C^* -algebra, we show that in our setting we automatically obtain coextensions. In fact, we show that in many cases, this phenomenon characterizes the complete Nevanlinna-Pick property. We also deduce operator theoretic dilation results which are in the spirit of work of Agler and several other authors. Moreover, we identify all boundary representations, compute the C^* -envelopes and determine hyperrigidity for certain analogues of the disc algebra. Finally, we extend these results to spaces of functions on homogeneous subvarieties of the ball.

1. INTRODUCTION

In this work, we study representations of multiplier algebras of reproducing kernel Hilbert spaces with the complete Nevanlinna-Pick property. Our motivating example is the classical disc algebra $A(\mathbb{D})$, which consists of those analytic functions on the open unit disc $\mathbb{D} \subset \mathbb{C}$ which extend to continuous functions on $\overline{\mathbb{D}}$. This algebra plays a pivotal role in the theory of contractions on Hilbert space, see [46] for a classical treatment. To make this role explicit, we must consider $A(\mathbb{D})$ in relation to the Hardy space $H^2(\mathbb{D})$, which is the Hilbert space of analytic functions on \mathbb{D} whose Taylor coefficients at the origin are square summable. Alternatively, the Hardy space may be described as the reproducing kernel Hilbert space on the unit disc with kernel

$$k(z, w) = \frac{1}{1 - z\overline{w}} \quad (z, w \in \mathbb{D}).$$

Then, for $\varphi \in A(\mathbb{D})$ we denote by M_φ the corresponding multiplication operator on $H^2(\mathbb{D})$. This identification allows us to regard $A(\mathbb{D})$ as an algebra of bounded linear operators on $H^2(\mathbb{D})$.

The (matrix-valued version of) von Neumann's inequality (see, for example, [40, Corollary 3.12]) shows that unital (completely) contractive representations of $A(\mathbb{D})$

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are in one-to-one correspondence with contractions on a Hilbert space. Specifically, every contraction $T \in B(\mathcal{H})$ gives rise to a unital completely contractive representation

$$A(\mathbb{D}) \rightarrow B(\mathcal{H}), \quad p \mapsto p(T) \quad (p \in \mathbb{C}[z]).$$

Abstract dilation theory in the form of Arveson's dilation theorem [8] (see also [40, Corollary 7.7]) shows that this representation dilates to a $*$ -representation π of $C^*(A(\mathbb{D})) \subset B(H^2(\mathbb{D}))$, in the sense that

$$p(T) = P_{\mathcal{H}} \pi(M_p)|_{\mathcal{H}}, \quad (p \in \mathbb{C}[z]).$$

The C^* -algebra $C^*(A(\mathbb{D}))$ is known as the Toeplitz algebra, whose representations are well understood. Indeed, M_z is the unilateral shift, hence $\pi(M_z)$ is an isometry. The Wold decomposition therefore implies that $\pi(M_z)$ is unitarily equivalent to $M_z^{(\alpha)} \oplus U$ for some cardinal α and some unitary operator U . In particular, we see that every contraction T dilates to an operator of the form $M_z^{(\alpha)} \oplus U$.

However, a stronger statement is true. Namely, one version of Sz.-Nagy's dilation theorem [46, Theorem 4.1], combined with the Wold decomposition, asserts that every contraction T does not merely dilate, but in fact coextends to an operator of the form $M_z^{(\alpha)} \oplus U$. That is, the dilation has the additional property that \mathcal{H} is invariant under $(M_z^{(\alpha)} \oplus U)^*$. In representation theoretic terms, this means that every unital completely contractive representation $\rho : A(\mathbb{D}) \rightarrow B(\mathcal{H})$ coextends to a $*$ -representation π of $C^*(A(\mathbb{D}))$, in the sense that

$$\rho(M_p)^* = \pi(M_p)^*|_{\mathcal{H}} \quad (p \in \mathbb{C}[z]).$$

Thus, when viewing $A(\mathbb{D})$ as a subalgebra of $B(H^2(\mathbb{D}))$, we conclude that its representations have a finer property than what is usually expected from general dilation theory.

The multivariate counterpart to this discussion takes place in the so-called Drury-Arveson space H_d^2 on the open unit ball $\mathbb{B}_d \subset \mathbb{C}^d$ [27, 38, 11], see also the survey [45]. This is the reproducing kernel Hilbert space on \mathbb{B}_d with kernel

$$k(z, w) = \frac{1}{1 - \langle z, w \rangle_{\mathbb{C}^d}}, \quad z, w \in \mathbb{B}_d.$$

Let \mathcal{A}_d be the norm closed subalgebra of $B(H_d^2)$ generated by the multiplication operators M_{z_1}, \dots, M_{z_d} . This algebra plays the role of the disc algebra in the study of *commuting row contractions*, that is, tuples $T = (T_1, \dots, T_d)$ of commuting operators such that $\sum_{j=1}^d T_j T_j^* \leq I$. As before, Arveson's dilation theorem shows that every unital completely contractive representation of \mathcal{A}_d dilates to a $*$ -representation π of the Toeplitz algebra $C^*(\mathcal{A}_d) \subset B(H_d^2)$ (and one can then use the known representation theory of the Toeplitz algebra to deduce an operator theoretic dilation result, see [11, Proposition 8.3]). Once again, however, something stronger is true. A dilation theorem due to Müller-Vasilescu [38] and Arveson [11] for commuting row contractions implies that unital completely contractive representations of \mathcal{A}_d in fact coextend to $*$ -representations of $C^*(\mathcal{A}_d)$.

Motivated by these examples, we ask:

Question 1. Let \mathcal{A} be a unital algebra of multipliers of a reproducing kernel Hilbert space, and let ρ be a unital completely contractive representation of \mathcal{A} . Does ρ coextend to a $*$ -representation of $C^*(\mathcal{A})$?

The discussion above shows that this question always has a positive answer for $A(\mathbb{D}) \subset B(H^2(\mathbb{D}))$ and, more generally, for $\mathcal{A}_d \subset B(H_d^2)$.

We saw above that the existence of an isometric dilation for a single contraction $T \in B(\mathcal{H})$ could be deduced from the inclusions

$$A(\mathbb{D}) \subset C^*(A(\mathbb{D})) \subset B(H^2(\mathbb{D})).$$

A different version of Sz.-Nagy's dilation [46, Theorem I.4.2] can be obtained by embedding $A(\mathbb{D})$ inside a different C^* -algebra. By the maximum modulus principle, if we denote by \mathbb{T} the unit circle, then we may regard $A(\mathbb{D})$ as a subalgebra of $C(\mathbb{T})$, via restriction to \mathbb{T} . In fact, the C^* -algebra $C(\mathbb{T})$ is the smallest C^* -algebra which contains a copy of $A(\mathbb{D})$ in a precise sense: it is the C^* -envelope of $A(\mathbb{D})$. An application of Arveson's dilation theorem in this setting yields a $*$ -representation π of $C(\mathbb{T})$ such that

$$p(T) = P_{\mathcal{H}}\pi(p)|_{\mathcal{H}}, \quad p \in \mathbb{C}[z].$$

Since the identity function z is unitary in $C(\mathbb{T})$, we thus obtain a unitary dilation of T . It is then natural to wonder whether a similar approach is possible for commuting row contractions and the algebra \mathcal{A}_d . Unfortunately this is not the case, as the C^* -algebra $C^*(\mathcal{A}_d)$ considered above is the C^* -envelope of \mathcal{A}_d . In fact, a stronger statement is true: the identity representation is a boundary representation for \mathcal{A}_d [11, Lemma 7.13] (we will review these notions in Subsection 3.5).

This motivates our second main question:

Question 2. Given an algebra of multipliers \mathcal{A} , what is its C^* -envelope? What are its boundary representations?

In particular, we will study Questions 1 and 2 for algebras of multipliers of reproducing kernel Hilbert spaces with the complete Nevanlinna-Pick property. These spaces mirror some of the fine structure of the classical Hardy space. We will recall the precise definition in Subsection 2.1. For now, we only mention that typical examples are the Hardy space itself, the Drury-Arveson space and also the Dirichlet space.

In the context of Question 1, we will show in Sections 4 and 5 that the “automatic coextension property”, which we witnessed for $A(\mathbb{D})$ and \mathcal{A}_d above, holds for many algebras of multipliers of complete Nevanlinna-Pick spaces. In fact, we will see in Section 9 that in many cases of interest, this phenomenon characterizes those spaces with the complete Nevanlinna-Pick property. As for Question 2, we show in Section 6 that the fact that the C^* -envelope of \mathcal{A}_d is $C^*(\mathcal{A}_d) \subset B(H_d^2)$ is the typical behaviour for many complete Nevanlinna-Pick spaces, and that the Hardy space is an exception in that regard.

We now proceed to state our main results more precisely and to explain how the paper is organized. Section 2 introduces the necessary background material and preliminary results about reproducing kernel Hilbert spaces. In particular, we recall the definition of unitarily invariant complete Nevanlinna-Pick spaces on the unit ball. To such a space \mathcal{H} , we associate an algebra of multipliers $A(\mathcal{H})$, which is the norm closure of the polynomial multipliers and serves as a generalization of the algebras $A(\mathbb{D})$ and \mathcal{A}_d mentioned above. Most of our results deal with these algebras. In Section 3, we recall the necessary background material about dilations.

In Section 4, we study a very concrete type of representations, namely inclusions of multiplier algebras of two reproducing kernel Hilbert spaces on the same set. In particular, we show the following result (Theorem 4.2).

Theorem 1.1. *Let \mathcal{H}_1 and \mathcal{H}_2 be two reproducing kernel Hilbert spaces on the same set X with reproducing kernels k_1, k_2 , respectively. Suppose that \mathcal{H}_1 is an irreducible complete Nevanlinna-Pick space. Then the following assertions are equivalent.*

- (i) *The inclusion $\text{Mult}(\mathcal{H}_1) \hookrightarrow \text{Mult}(\mathcal{H}_2)$ is completely contractive.*
- (ii) *The inclusion $\text{Mult}(\mathcal{H}_1) \hookrightarrow \text{Mult}(\mathcal{H}_2)$ coextends to a $*$ -representation of $B(\mathcal{H}_1)$.*
- (iii) *$\text{Mult}(\mathcal{H}_1) \subset \text{Mult}(\mathcal{H}_2)$ and there exists an auxiliary Hilbert space \mathcal{E} such that $M_\varphi^{\mathcal{H}_2}$ coextends to $M_\varphi^{\mathcal{H}_1} \otimes I_{\mathcal{E}}$ for every $\varphi \in \text{Mult}(\mathcal{H}_1)$.*
- (iv) *k_2/k_1 is positive semi-definite on X .*

This result is in fact a combination of previously known theorems of Douglas-Misra-Sarkar [24], Kumari-Sarkar-Sarkar [36] and a result from [34], but it appears to be new in this form. Moreover, it shows that Question 1 has a positive answer for these particular representations. In fact, we show that the validity of the implication (i) \Rightarrow (iv) characterizes those reproducing kernel Hilbert spaces with the complete Nevanlinna-Pick property (Theorem 4.4), which gives the first indication that this property indeed plays a key role.

In Section 5, we consider representations of the algebras $A(\mathcal{H})$ associated to unitarily invariant complete Nevanlinna-Pick spaces on the unit ball. We describe all unital completely contractive representations of these algebras and give a positive answer to Question 1 in this case. To achieve this, we will make use of a general machinery for studying coextensions which has been developed over the years by Agler [1], Agler-McCarthy [3] and Ambrozie-Engliš-Müller [5]. The main result of that section is summarized in the following theorem, which is a combination of Theorem 5.4 and Corollary 5.5.

Theorem 1.2. *Let \mathcal{H} be a unitarily invariant complete Nevanlinna-Pick space on the unit ball with reproducing kernel k . Let $T = (T_1, \dots, T_d)$ be a commuting tuple of operators on a Hilbert space \mathcal{E} . Then, the following assertions are equivalent.*

- (i) *The tuple T satisfies $1/k(T, T^*) \geq 0$.*
- (ii) *There exists a unital completely contractive homomorphism*

$$\rho : A(\mathcal{H}) \rightarrow B(\mathcal{E})$$

such that $\rho(M_z) = T$.

In this case, the map ρ is necessarily unique and coextends to a $$ -representation of $C^*(A(\mathcal{H}))$. In particular, every unital completely contractive representation of $A(\mathcal{H})$ coextends to a $*$ -representation of $C^*(A(\mathcal{H}))$.*

The precise definition of $1/k(T, T^*)$ can be found in Section 5. For now, we simply mention that if $\mathcal{H} = H_d^2$, then

$$1/k(T, T^*) = 1 - \sum_{j=1}^d T_j T_j^*$$

so $1/k(T, T^*) \geq 0$ if and only if T is a row contraction.

As a consequence, we establish the following dilation theorem (Theorem 5.6), which is very much in the spirit of [1, 3, 5] and in addition contains an automatic coextension statement. It extends the Müller-Vasilescu and Arveson dilation theorem [38, 11]. Recall that a spherical unitary is a tuple of commuting normal operators whose joint spectrum is contained in the unit sphere.

Theorem 1.3. *Let \mathcal{H} be a unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d with reproducing kernel*

$$k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n,$$

where $a_0 = 1$ and $a_n > 0$ for all $n \in \mathbb{N}$. Suppose that $\lim_{n \rightarrow \infty} a_n/a_{n+1} = 1$. Let $T = (T_1, \dots, T_d)$ be a commuting tuple of operators on a Hilbert space. Denote by $M_z = (M_{z_1}, \dots, M_{z_d})$ the tuple of operators of multiplication by the coordinate functions. Then the following assertions are equivalent.

- (i) The tuple T satisfies $1/k(T, T^*) \geq 0$.
- (ii) The tuple T coextends to $M_z^{(\kappa)} \oplus U$ for some cardinal κ and a spherical unitary U .
- (iii) The tuple T dilates to $M_z^{(\kappa)} \oplus U$ for some cardinal κ and a spherical unitary U .

The regularity condition on the coefficients a_n is discussed in Subsection 2.6. It is satisfied for many spaces of interest including the Hardy space, the Drury-Arveson space and the Dirichlet space.

In Section 6, we study Question 2 for the algebras $A(\mathcal{H})$. More precisely, we identify all the boundary representations (Theorem 6.2), compute the C^* -envelope (Corollary 6.4), and determine hyperrigidity (Theorem 6.3) for the algebras $A(\mathcal{H})$, where \mathcal{H} is a unitarily invariant complete Nevanlinna-Pick space which satisfies an additional regularity assumption. (We will review these notions in Subsection 3.5.) These results make use of earlier results by Guo-Hu-Xu [31] and can be summarized as follows.

Theorem 1.4. *Let \mathcal{H} be a unitarily invariant complete Nevanlinna-Pick space on the unit ball with reproducing kernel*

$$k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n$$

where $a_0 = 1$ and $a_n > 0$ for all $n \in \mathbb{N}$. Suppose that $\lim_{n \rightarrow \infty} a_n/a_{n+1} = 1$ and that \mathcal{H} is not the Hardy space $H^2(\mathbb{D})$.

- (a) If $\sum_{n=0}^{\infty} a_n = \infty$, then the boundary representations for $A(\mathcal{H})$ are precisely the characters of evaluation at a point on the sphere, along with the identity representation of $C^*(A(\mathcal{H}))$. Furthermore, the algebra $A(\mathcal{H})$ is hyperrigid.
- (b) If $\sum_{n=0}^{\infty} a_n < \infty$, then the identity representation of $C^*(A(\mathcal{H}))$ is the only boundary representation for $A(\mathcal{H})$. In particular, the algebra $A(\mathcal{H})$ is not hyperrigid.

In particular, the C^* -envelope of $A(\mathcal{H})$ is $C^*(A(\mathcal{H})) \subset B(\mathcal{H})$ in each case.

It is worth pointing out that all the C^* -algebras $C^*(A(\mathcal{H}))$ are in fact unitarily equivalent (Proposition 6.5). We now briefly address the situation where \mathcal{H} is the Hardy space. In that case, the algebra $A(\mathcal{H})$ coincides with the disc algebra $A(\mathbb{D})$, the boundary representations of which are well known to be the characters of evaluation at a point in \mathbb{T} . In particular, the C^* -envelope of $A(\mathbb{D})$ is $C(\mathbb{T})$, and $A(\mathbb{D})$ is not hyperrigid inside $C^*(A(\mathbb{D})) \subset B(H^2(\mathbb{D}))$.

In Sections 7 and 8, we extend the results mentioned above to quotients of $A(\mathcal{H})$ by homogeneous polynomial ideals. In this setting, Arveson's famous essential normality conjecture plays an important role, and we review some relevant background on this topic in Section 7. Section 8 contains the desired generalizations. The results on boundary representations and hyperrigidity therein are related to some of the recent contributions of Kennedy and Shalit [35] on the Drury-Arveson space. Moreover, they fit in well with the recent interest in operator algebras associated to geometric varieties [22, 23, 34].

In the final Section 9, we show that the automatic coextension property observed in Theorem 1.2 is in fact equivalent to the complete Nevanlinna-Pick property for a familiar class of reproducing kernel Hilbert spaces on the unit ball. The main result of this section is the following theorem (Theorem 9.4), whose proof uses some of the machinery on homogeneous ideals developed in Section 8.

Theorem 1.5. *Let \mathcal{H} be a unitarily invariant space on the unit ball with reproducing kernel*

$$k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n,$$

where $a_0 = 1$ and $a_n > 0$ for all $n \in \mathbb{N}$. Suppose that $\lim_{n \rightarrow \infty} a_n/a_{n+1} = 1$. If every unital completely contractive representation of $A(\mathcal{H})$ coextends to a $*$ -representation of $C^*(A(\mathcal{H}))$, then \mathcal{H} is a complete Nevanlinna-Pick space.

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2. PRELIMINARIES ON REPRODUCING KERNEL HILBERT SPACES

2.1. Kernels and Nevanlinna-Pick spaces. Let \mathcal{H} be a reproducing kernel Hilbert space on a set X . Background material on reproducing kernel Hilbert spaces can be found in [41] and [4]. Let k denote the reproducing kernel of X , which is positive semi-definite in the sense that for any finite set of points $x_1, \dots, x_n \in X$, the $n \times n$ matrix $[k(x_i, x_j)]_{i,j=1}^n$ is positive semi-definite. The central object of our investigations is not the space \mathcal{H} itself as much as its *multiplier algebra*

$$\text{Mult}(\mathcal{H}) = \{\varphi : X \rightarrow \mathbb{C} : \varphi f \in \mathcal{H} \text{ for all } f \in \mathcal{H}\}.$$

An application of the closed graph theorem shows that for $\varphi \in \text{Mult}(\mathcal{H})$, the associated multiplication operator $M_\varphi^{\mathcal{H}}$ is bounded on \mathcal{H} . Unless otherwise noted, we will always assume that \mathcal{H} has no common zeros, or equivalently, that $k(w, w) \neq 0$ for all $w \in X$. In this case, the assignment $\varphi \mapsto M_\varphi^{\mathcal{H}}$ is injective, so we may regard $\text{Mult}(\mathcal{H})$ as a subalgebra of $B(\mathcal{H})$ in this way. When the space \mathcal{H} is clear from context, we simply write M_φ instead of $M_\varphi^{\mathcal{H}}$.

Most of the reproducing kernel Hilbert spaces we are interested in will be assumed to be *irreducible* in the sense that $k(z, w) \neq 0$ for all $z, w \in X$, and $k(\cdot, w)$ and $k(\cdot, v)$ are linearly independent if $v \neq w$ (see [4, Section 7.1]). Moreover, it will sometimes be convenient to assume that the reproducing kernel k is *normalized at* $x_0 \in X$, which means that $k(x, x_0) = 1$ for all $x \in X$. A reproducing kernel Hilbert space \mathcal{H} is said to be a *complete Nevanlinna-Pick space* if for all positive integers

r, n , every collection of points z_1, \dots, z_n in X and every choice of $r \times r$ complex matrices W_1, \dots, W_n , the non-negativity of the block matrix

$$\left[k(z_i, z_j)(I - W_i W_j^*) \right]_{i,j=1}^n$$

is sufficient for the existence of a matrix-valued multiplier Φ [4, Section 2.8] with $\|\Phi\| \leq 1$ such that

$$\Phi(z_i) = W_i, \quad i = 1, \dots, n.$$

Classical examples of complete Nevanlinna-Pick spaces include the Hardy space $H^2(\mathbb{D})$ and the Drury-Arveson space H_d^2 mentioned in the introduction. In general, irreducible complete Nevanlinna-Pick spaces are characterized in terms of positivity of certain matrices by a theorem of McCullough and Quiggin [43, 37]. A related characterization is due to Agler and McCarthy [2]. We refer the reader to [4, Chapter 7] for a comprehensive account on the characterization of the complete Nevanlinna-Pick spaces.

2.2. Unitarily invariant spaces. Although we will spend some time working at the level of generality described above, some of our finer results are obtained for a more concrete subclass of reproducing kernel Hilbert spaces, which we now introduce. Let $\mathbb{B}_d \subset \mathbb{C}^d$ denote the open unit ball. A *unitarily invariant space* on \mathbb{B}_d is a reproducing kernel Hilbert space with a kernel of the form

$$k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n$$

for some sequence of strictly positive coefficients $\{a_n\}_n$ such that $a_0 = 1$. Since $a_0 = 1$, the kernel k is normalized at 0. It is well known that the monomials $\{z^\alpha\}_{\alpha \in \mathbb{N}^d \setminus \{0\}}$ form an orthogonal basis for \mathcal{H} and that

$$\|z^\alpha\|_{\mathcal{H}}^2 = \frac{1}{a_{|\alpha|}} \frac{\alpha!}{|\alpha|!}.$$

We use the usual multi-index notation: given $\alpha \in \mathbb{N}^d \setminus \{0\}$ we have

$$\alpha = (\alpha_1, \dots, \alpha_d), \quad |\alpha| = \alpha_1 + \dots + \alpha_d, \quad \alpha! = \alpha_1! \cdots \alpha_d!$$

and

$$z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}.$$

See [30, Section 4] for more detail. Here we also feel compelled to point out that in view of [2], the case where d is an infinite cardinal number is important, however in this paper we will always assume that d is a finite positive integer.

By a *unitarily invariant complete Nevanlinna-Pick space* on the ball, we mean a unitarily invariant space \mathcal{H} on \mathbb{B}_d with kernel $k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n$ having the following additional properties:

- (1) the power series $\sum_{n=0}^{\infty} a_n t^n$ has radius of convergence 1, and
- (2) \mathcal{H} is an irreducible complete Nevanlinna-Pick space.

The radius of convergence of the power series appearing in condition (1) is at least 1, as k is defined on $\mathbb{B}_d \times \mathbb{B}_d$. Demanding that it be exactly 1 guarantees that the functions in \mathcal{H} do not all extend to analytic functions on a ball of radius greater than 1. This condition is closely related to the notion of algebraic consistency, see [34, Section 5] and in particular Lemma 5.3 therein. Finally, in relation to property (2), the characterization of the complete Nevanlinna-Pick property takes

on a particularly simple form for unitarily invariant spaces, and we record it below for convenience. It is essentially [4, Lemma 7.33]. The precise statement can also be found in [34, Lemma 2.3].

Lemma 2.1. *Let \mathcal{H} be a unitarily invariant space on \mathbb{B}_d with reproducing kernel*

$$k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n.$$

Then \mathcal{H} is an irreducible complete Nevanlinna-Pick space if and only if

$$1 - \frac{1}{k(z, w)} = \sum_{n=1}^{\infty} b_n \langle z, w \rangle^n \quad (z, w \in \mathbb{B}_d)$$

for some sequence $\{b_n\}_n$ of non-negative numbers.

Observe that the sequence $\{b_n\}_n$ above is in fact uniquely determined by the reproducing kernel k .

2.3. Bounded and unbounded kernels. When working with unitarily invariant spaces, we will find it convenient to distinguish two cases, depending on the summability of the sequence $\{a_n\}_n$, or equivalently, on the boundedness of the reproducing kernel.

- The first case corresponds to $\sum_{n=0}^{\infty} a_n = \infty$. Then, for every ζ in the unit sphere, we have $k(w, w) \rightarrow \infty$ as $w \rightarrow \zeta$. As a consequence, the space \mathcal{H} contains a function which is unbounded near ζ , and the open unit ball is the natural domain of definition for the functions in \mathcal{H} . Indeed, this can be made rigorous by saying that \mathcal{H} is an algebraically consistent space on \mathbb{B}_d (see [34, Lemma 5.3]), but we will not require this more precise statement.

- The second case corresponds to $\sum_{n=0}^{\infty} a_n < \infty$. Then, the reproducing kernel k extends to a continuous function \hat{k} on $\overline{\mathbb{B}_d} \times \overline{\mathbb{B}_d}$. It follows that all functions in \mathcal{H} extend to continuous functions on $\overline{\mathbb{B}_d}$, and thus \mathcal{H} can be viewed as a reproducing kernel Hilbert space on $\overline{\mathbb{B}_d}$ whose kernel is \hat{k} . Indeed, if $\hat{\mathcal{H}}$ denotes the reproducing kernel Hilbert space on $\overline{\mathbb{B}_d}$ with kernel \hat{k} , then $\hat{\mathcal{H}}|_{\mathbb{B}_d} = \mathcal{H}$. Moreover, since \hat{k} is continuous on $\overline{\mathbb{B}_d} \times \overline{\mathbb{B}_d}$, the functions in $\hat{\mathcal{H}}$ are continuous on $\overline{\mathbb{B}_d}$, so the coisometric restriction map $\hat{\mathcal{H}} \rightarrow \mathcal{H}$ (see [6, Section I.5]) is injective, and hence unitary. In this setting, the natural domain of definition for the functions in \mathcal{H} is the closed unit ball (see [34, Lemma 5.3] for a more precise version of this statement).

2.4. Examples. The class of unitarily invariant complete Nevanlinna-Pick spaces on the ball contains many widely studied spaces. For instance, for $s \leq 0$ consider the kernel

$$\sum_{n=0}^{\infty} (n+1)^s \langle z, w \rangle^n$$

on the unit ball. Then, the associated reproducing kernel Hilbert space $\mathcal{H}_s(\mathbb{B}_d)$ is a unitarily invariant complete Nevanlinna-Pick space (see [4, Corollary 7.41]). If $-1 \leq s \leq 0$, the space $\mathcal{H}_s(\mathbb{B}_d)$ falls into the first category discussed in Subsection 2.3, and if $s < -1$ it belongs to the second category.

If $d = 1$, then these are spaces on the unit disc \mathbb{D} , which we simply denote by \mathcal{H}_s . The scale of spaces $\{\mathcal{H}_s : s \leq 0\}$ contains in particular the Hardy space ($s = 0$) and the Dirichlet space ($s = -1$), the latter of which is the source of many deep

problems and the target of intense current research; see for instance the book [28] and the references therein. For every $d \in \mathbb{N}$, the space $\mathcal{H}_0(\mathbb{B}_d)$ is the Drury-Arveson space H_d^2 , which was mentioned in the introduction.

A related class of examples are the Besov-Sobolev spaces \mathcal{K}_σ on \mathbb{B}_d with reproducing kernel

$$\frac{1}{(1 - \langle z, w \rangle)^\sigma},$$

where $\sigma \in (0, 1]$. It is well known that \mathcal{K}_σ and $\mathcal{H}_{\sigma-1}$ coincide as vector spaces and that their norms are equivalent.

2.5. The algebras $A(\mathcal{H})$. If \mathcal{H} is a unitarily invariant space on the ball such that the coordinate functions z_1, \dots, z_d are multipliers, then we define

$$A(\mathcal{H}) = \overline{\mathbb{C}[z_1, \dots, z_d]}^{\|\cdot\|} \subset \text{Mult}(\mathcal{H}).$$

That is, the operator algebra $A(\mathcal{H})$ is the norm closure of the polynomials inside the multiplier algebra $\text{Mult}(\mathcal{H})$. If \mathcal{H} is a unitarily invariant complete Nevanlinna-Pick space, then the coordinate functions are automatically multipliers (see the discussion at the beginning of [30, Section 4]), so that $A(\mathcal{H})$ is always defined in this case. Algebras of the type $A(\mathcal{H})$ will be one of our main objects of interest throughout the paper.

Note that if \mathcal{H} is the Hardy space $H^2(\mathbb{D})$, then $A(\mathcal{H})$ is the classical disc algebra $A(\mathbb{D})$. If \mathcal{H} is the Drury-Arveson space H_d^2 , then $A(\mathcal{H})$ is the algebra \mathcal{A}_d discussed in the introduction, which plays a central role in multivariable operator theory. Indeed, spurred on by Popescu's [42] and Arveson's [11] seminal works, this algebra and its non-commutative counterpart have received constant and considerable attention over the years (see for instance [21, 22, 19, 18]).

2.6. Regular spaces and Toeplitz algebras. As mentioned in the introduction, we will sometimes require an additional property of a unitarily invariant space on \mathbb{B}_d with reproducing kernel

$$k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n,$$

namely that

$$\lim_{n \rightarrow \infty} a_n / a_{n+1} = 1.$$

We will call such a space *regular*. This assumption is fairly common in the study of unitarily invariant spaces (see for example [30, Section 4] or [34, Proposition 8.5]). All the classical spaces considered in Subsection 2.4 enjoy this property. Since we usually assume that the power series $\sum_{n=0}^{\infty} a_n t^n$ has radius of convergence 1, the quantity $\lim_{n \rightarrow \infty} a_n / a_{n+1}$, if it exists, is necessarily 1. Thus, we regard this extra condition as a regularity assumption on the kernel, hence the terminology. If \mathcal{H} is regular, then the polynomials are automatically multipliers on \mathcal{H} (see [31, Corollary 4.4 (1)]). In our context, the usefulness of the regularity assumption stems from the following result, which is [31, Proposition 4.3 and Theorem 4.6]. Throughout, we denote the ideal of compact operators on a Hilbert space \mathcal{E} by $\mathcal{K}(\mathcal{E})$ and the unit sphere (the boundary of the unit ball) by $\partial \mathbb{B}_d$.

Theorem 2.2 (Guo-Hu-Xu). *Let \mathcal{H} be a regular unitarily invariant space on \mathbb{B}_d . Then, the operator $I - \sum_{j=1}^d M_{z_j} M_{z_j}^*$ is compact, and there is a short exact sequence*

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow C^*(A(\mathcal{H})) \longrightarrow C(\partial\mathbb{B}_d) \longrightarrow 0,$$

where the first map is the inclusion and the second map sends M_{z_j} to the coordinate function z_j for each $1 \leq j \leq d$.

3. PRELIMINARIES ON DILATIONS

In this section, we recall a few basic notions from dilation theory. For background material, the reader is referred to the book [40].

3.1. Dilations of operator tuples. Let $T = (T_1, \dots, T_d)$ be a tuple of commuting operators on a Hilbert space \mathcal{H} , let \mathcal{L} be a Hilbert space which contains \mathcal{H} as a closed subspace and let $S = (S_1, \dots, S_d)$ be a tuple of commuting operators on \mathcal{L} . We say that S is a *dilation* of T , or that T *dilates* to S , if

$$p(T) = P_{\mathcal{H}} p(S)|_{\mathcal{H}} \quad \text{for all } p \in \mathbb{C}[z_1, \dots, z_d].$$

By a theorem of Sarason (see, for example [40, Theorem 7.6]), this happens if and only if \mathcal{L} admits a decomposition $\mathcal{L} = \mathcal{L}_- \oplus \mathcal{H} \oplus \mathcal{L}_+$ with respect to which we can write

$$S_k = \begin{bmatrix} * & 0 & 0 \\ * & T_k & 0 \\ * & * & * \end{bmatrix} \quad (k = 1, \dots, d).$$

We say that S is an *extension* of T , or that T *extends* to S , if the space \mathcal{L}_+ above is absent. Likewise, we say that S is a *coextension* of T , or that T *coextends* to S , if the space \mathcal{L}_- is absent, that is, if

$$S_k = \begin{bmatrix} T_k & 0 \\ * & * \end{bmatrix} \quad (k = 1, \dots, d).$$

Clearly S^* is an extension of T^* if and only if S is a coextension of T .

3.2. Dilations of representations. While dilations of operators are our main motivation, it will be beneficial to adopt a more operator algebraic point of view. Let \mathcal{A} be a unital operator algebra. A *representation* of \mathcal{A} is a unital homomorphism

$$\rho : \mathcal{A} \rightarrow B(\mathcal{H})$$

for some Hilbert space \mathcal{H} . Let $\mathcal{L} \supset \mathcal{H}$ be a larger Hilbert space, let $\mathcal{B} \supset \mathcal{A}$ be a larger operator algebra and let $\sigma : \mathcal{B} \rightarrow B(\mathcal{L})$ be another representation. We say that ρ *dilates* to σ if

$$P_{\mathcal{H}} \sigma(a)|_{\mathcal{H}} = \rho(a) \quad \text{for all } a \in \mathcal{A}.$$

By the aforementioned result of Sarason, this happens if and only if \mathcal{L} admits a decomposition $\mathcal{L} = \mathcal{L}_- \oplus \mathcal{H} \oplus \mathcal{L}_+$ with respect to which we can write

$$\sigma(a) = \begin{bmatrix} * & 0 & 0 \\ * & \rho(a) & 0 \\ * & * & * \end{bmatrix} \quad \text{for all } a \in \mathcal{A}.$$

We say that ρ *extends* to σ if \mathcal{L}_+ is absent, and ρ *coextends* to σ if \mathcal{L}_- is absent.

Of particular interest is the case when \mathcal{B} is a unital C^* -algebra which contains \mathcal{A} . In this case, we say that ρ *admits a \mathcal{B} -dilation* if ρ dilates to a $*$ -representation of \mathcal{B} .

Similarly, we say that ρ admits a \mathcal{B} -coextension if ρ coextends to a $*$ -representation of \mathcal{B} .

Let $n \in \mathbb{N}$ and denote by $M_n(\mathcal{A})$ the algebra of $n \times n$ matrices with entries from \mathcal{A} . A linear map $\psi : \mathcal{A} \rightarrow B(\mathcal{H})$ induces another linear map $\psi^{(n)} : M_n(\mathcal{A}) \rightarrow M_n(B(\mathcal{H}))$, defined by applying ψ entry-wise. Then, ψ is said to be *completely contractive* (respectively, *completely isometric*) if the induced map $\psi^{(n)}$ is contractive (respectively, isometric) for every $n \in \mathbb{N}$. Using these ideas, Arveson [8] characterized abstractly when a representation of \mathcal{A} admits a $C^*(\mathcal{A})$ -dilation (see also [40, Corollary 7.7]).

Theorem 3.1 (Arveson). *Let \mathcal{B} be a unital C^* -algebra, let $\mathcal{A} \subset \mathcal{B}$ be a unital subalgebra of \mathcal{B} and let $\rho : \mathcal{A} \rightarrow B(\mathcal{H})$ be a representation of \mathcal{A} . Then ρ admits a \mathcal{B} -dilation if and only if it is completely contractive.*

From an operator theoretic point of view, it may seem more natural to rephrase results about coextensions in terms of extensions of the adjoints. In our case, however, the operator algebraic formulation makes it more convenient to work with the coextensions directly.

3.3. Coextensions of unital completely positive maps. An *operator system* is a unital self-adjoint subspace of a unital C^* -algebra. A linear map ψ between operator systems is said to be *positive* if it maps positive elements to positive elements, and *completely positive* if the induced map $\psi^{(n)}$ is positive for every $n \in \mathbb{N}$.

We will occasionally require a slightly more detailed description of the dilation in Theorem 3.1. If \mathcal{A} is a unital subalgebra of a unital C^* -algebra \mathcal{B} and $\rho : \mathcal{A} \rightarrow B(\mathcal{H})$ is a unital completely contractive representation, then ρ extends uniquely to a unital completely positive linear map $\tilde{\rho} : \mathcal{A} + \mathcal{A}^* \rightarrow B(\mathcal{H})$ [40, Proposition 3.5]. Arveson's extension theorem [40, Theorem 7.5] implies that $\tilde{\rho}$ extends to a unital completely positive linear map on \mathcal{B} , and Stinespring's dilation theorem [40, Theorem 4.1] then yields the desired \mathcal{B} -dilation of ρ .

To obtain coextensions as opposed to mere dilations, we will use a device of Agler, which has its roots in his work on hereditary polynomials [1]. In our context, the relevant statement is the following lemma (see [1, Theorem 1.5] and its proof).

Lemma 3.2 (Agler). *Let \mathcal{B} be a unital C^* -algebra, let $\psi : \mathcal{B} \rightarrow B(\mathcal{H})$ be a unital completely positive linear map and let $\pi : \mathcal{B} \rightarrow B(\mathcal{L})$ be a $*$ -representation such that*

$$\psi(T) = P_{\mathcal{H}}\pi(T)|_{\mathcal{H}}$$

for every $T \in \mathcal{B}$. For each $T \in \mathcal{B}$, the following assertions are equivalent:

- (i) $\psi(TT^*) = \psi(T)\psi(T)^*$,
- (ii) \mathcal{H} is coinvariant for $\pi(T)$.

It should be noted that given a unital completely positive map ψ as in the lemma above, there always exists a $*$ -representation π with

$$\psi(T) = P_{\mathcal{H}}\pi(T)|_{\mathcal{H}} \quad (T \in \mathcal{B}).$$

Indeed, this is the content of Stinespring's dilation theorem.

3.4. Representations of C^* -algebras containing the compact operators.

We will frequently make use of the following standard fact about representations of C^* -algebras containing the compact operators.

Lemma 3.3. *Let $\mathcal{B} \subset B(\mathcal{H})$ be a unital C^* -algebra which contains the ideal \mathcal{K} of compact operators on \mathcal{H} , and let π be a unital $*$ -representation of \mathcal{B} . Then π splits as a direct sum $\pi = \pi_1 \oplus \pi_2$, where π_1 is unitarily equivalent to a multiple of the identity representation and π_2 annihilates \mathcal{K} .*

Proof. This follows from the discussion preceding Theorem I.3.4 in [10] and the fact that every non-degenerate representation of \mathcal{K} is unitarily equivalent to a multiple of the identity representation. \square

3.5. Boundary representations and the C^* -envelope. Let $\mathcal{A} \subset B(\mathcal{H})$ be a unital operator algebra and $\psi : \mathcal{A} \rightarrow B(\mathcal{E})$ be a unital completely contractive linear map. As explained above, there exists a unital completely positive linear map $\pi : C^*(\mathcal{A}) \rightarrow B(\mathcal{E})$ which agrees with ψ on \mathcal{A} . We say that ψ has the *unique extension property* if there is a unique such extension, and this extension is a $*$ -homomorphism. An irreducible $*$ -representation π of $C^*(\mathcal{A})$ is said to be a *boundary representation for \mathcal{A}* if $\pi|_{\mathcal{A}}$ has the unique extension property. This idea was introduced by Arveson [8] in analogy with the classical Choquet boundary of a uniform algebra. It follows from [8, Theorem 2.1.2] that this notion does not depend on the concrete embedding of \mathcal{A} into $B(\mathcal{H})$, and is thus a completely isometric invariant of \mathcal{A} . The algebra \mathcal{A} is said to be *hypermorphic* [15] if the restriction $\pi|_{\mathcal{A}}$ has the unique extension property for every unital $*$ -representation π of $C^*(\mathcal{A})$.

Any operator algebra \mathcal{A} has sufficiently many boundary representations in the sense that there exists a set of boundary representations whose direct sum is completely isometric. This was first proved by Arveson [14] in the separable case, and later established in full generality by Davidson and Kennedy [20].

In general, there are many C^* -algebras which are generated by a completely isometric copy of a unital operator algebra \mathcal{A} , and accordingly it is natural to look for the smallest one in the following precise sense. The *C^* -envelope of \mathcal{A}* is a C^* -algebra $C_e^*(\mathcal{A})$ together with a unital completely isometric linear map $\iota : \mathcal{A} \rightarrow C_e^*(\mathcal{A})$ with the following two properties:

- (1) $C_e^*(\mathcal{A}) = C^*(\iota(\mathcal{A}))$,
- (2) whenever $j : \mathcal{A} \rightarrow \mathcal{B}$ is another unital completely isometric linear map and $\mathcal{B} = C^*(j(\mathcal{A}))$, then there exists a surjective $*$ -homomorphism $\pi : \mathcal{B} \rightarrow C_e^*(\mathcal{A})$ such that $\pi \circ j = \iota$.

It follows from [8, Theorem 2.1.2] that if ι is a direct sum of boundary representations for \mathcal{A} which is completely isometric on \mathcal{A} , then $C^*(\iota(\mathcal{A}))$ is the C^* -envelope of \mathcal{A} (see also [7]). In particular, if the identity representation of $C^*(\mathcal{A})$ on $B(\mathcal{H})$ is a boundary representation for \mathcal{A} , then $C_e^*(\mathcal{A}) = C^*(\mathcal{A})$. Even before it was known that there are sufficiently many boundary representations, these ideas were used by Dritschel and McCullough to construct the C^* -envelope [26], but the existence of such an object was proved yet earlier using different methods in [33].

3.6. A sufficient condition for the unique extension property. While we know that boundary representations are always plentiful as discussed above, determining when a given unital completely contractive representation has the unique extension property remains a difficult task in general. There are some instances

however where this is possible, and we proceed to describe one that will be useful for us. For that purpose, recall that a commuting family $T = (T_1, T_2, \dots)$ of operators on a Hilbert space is said to be a *row contraction* if

$$\sum_n T_n T_n^* \leq I,$$

where the series converges in the strong operator topology. Moreover, a commuting tuple $U = (U_1, U_2, \dots)$ of operators on a Hilbert space is said to be a *spherical unitary* if each U_n is normal and

$$\sum_n U_n U_n^* = I,$$

where the series converges in the strong operator topology as well. The following result follows from a theorem of Richter and Sundberg [44] and is a generalization of [35, Corollary 2.2]; the reader may also want to compare with [8, Theorem 3.1.2].

Theorem 3.4. *Let $T = (T_1, T_2, \dots)$ be a commuting row contraction and let \mathcal{A}_T be the unital norm closed operator algebra generated by $\{T_n : n \geq 1\}$. Let $\rho : \mathcal{A}_T \rightarrow B(\mathcal{H})$ be a unital completely contractive linear map such that $(\rho(T_1), \rho(T_2), \dots)$ is a spherical unitary. Then ρ has the unique extension property.*

Proof. Let $\hat{\rho} : C^*(\mathcal{A}_T) \rightarrow B(\mathcal{H})$ be a unital completely positive extension of ρ , and let $\pi : C^*(\mathcal{A}_T) \rightarrow B(\mathcal{L})$ be a Stinespring dilation of $\hat{\rho}$. We will show that \mathcal{H} is reducing for $\pi(C^*(\mathcal{A}_T))$, which implies that $\hat{\rho}$ is itself a $*$ -homomorphism and thus the unique completely positive extension of ρ .

First, we may use the Schwarz inequality for completely positive maps to obtain that $\rho(T_n)\rho(T_n)^* \leq \hat{\rho}(T_n T_n^*)$ for every n . Thus we have

$$I = \sum_n \rho(T_n)\rho(T_n)^* \leq \sum_n \hat{\rho}(T_n T_n^*) \leq I,$$

where the last inequality is due to the fact that T is a row contraction while $\hat{\rho}$ is unital and completely positive. Thus, equality holds throughout, so $\hat{\rho}(T_n)\hat{\rho}(T_n)^* = \hat{\rho}(T_n T_n^*)$ for all n . Lemma 3.2 now implies that \mathcal{H} is coinvariant for each $\pi(T_n)$, hence

$$\pi(T_n)^*|_{\mathcal{H}} = \rho(T_n)^*$$

for each n . We see that the commuting tuple $(\pi(T_1), \pi(T_2), \dots)$ is a row contraction, and $(\pi(T_1)^*, \pi(T_2)^*, \dots)$ is an extension of the spherical unitary $(\rho(T_1)^*, \rho(T_2)^*, \dots)$. It follows from [44, Theorem 2.2] that \mathcal{H} is also invariant for $\pi(T_n)$ for each n (although that theorem is only stated for finite tuples of operators, it is clear that the proof extends verbatim to the case of infinite tuples). Therefore, \mathcal{H} reduces each $\pi(T_n)$, and hence reduces $\pi(C^*(\mathcal{A}_T))$. \square

We remark that the proof establishes in fact the following more general statement. Let $T = (T_1, T_2, \dots)$ be a commuting row contraction and let \mathcal{S}_T denote the operator system generated by T . If $\rho : \mathcal{S}_T \rightarrow B(\mathcal{H})$ is a unital completely positive map such that $(\rho(T_1), \rho(T_2), \dots)$ is a spherical unitary, then ρ has the unique extension property.

4. COMPLETELY CONTRACTIVE INCLUSIONS OF MULTIPLIER ALGEBRAS

The aim of the next two sections is to address Question 1 for algebras of multipliers of reproducing kernel Hilbert spaces. Let \mathcal{H}_1 and \mathcal{H}_2 be two reproducing kernel Hilbert spaces on the same set X with respective kernels k_1 and k_2 . In this section, we aim to determine when $\text{Mult}(\mathcal{H}_1)$ is contained completely contractively in $\text{Mult}(\mathcal{H}_2)$. When this occurs, the inclusion may be regarded as a particular representation of $\text{Mult}(\mathcal{H}_1)$.

There is a basic sufficient condition for the inclusion to be completely contractive, namely that k_2/k_1 be positive semi-definite on X . Indeed, this is an easy consequence of the Schur product theorem (see, for instance, the implication (ii) \Rightarrow (i) in [34, Corollary 3.5]). Interestingly, this positivity condition on the kernels is also sufficient for the inclusion to admit a special kind of $B(\mathcal{H}_1)$ -coextension. This non-trivial fact is known, see [24, Theorem 2] and [36, Corollary 6.2]. Since the setting of these papers is slightly different, we provide a sketch of the proof of a statement adapted to our purposes.

Theorem 4.1. *Let \mathcal{H}_1 and \mathcal{H}_2 be reproducing kernel Hilbert spaces on the same set X with respective kernels k_1, k_2 . Assume that k_1 is non-vanishing and that k_2/k_1 is positive semi-definite. Then $\text{Mult}(\mathcal{H}_1) \subset \text{Mult}(\mathcal{H}_2)$ and there exists a Hilbert space \mathcal{E} along with an isometric map $V : \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{E}$ with the property that*

$$V^*(M_\varphi^{\mathcal{H}_1} \otimes I_{\mathcal{E}}) = M_\varphi^{\mathcal{H}_2} V^*,$$

for every $\varphi \in \text{Mult}(\mathcal{H}_1)$.

Proof. Since k_2/k_1 is positive, a well-known factorization theorem for positive semi-definite kernels (see [4, Theorem 2.53]) implies that there exists a Hilbert space \mathcal{E} and a function $f : X \rightarrow \mathcal{E}$ such that

$$k_2(x, y) = k_1(x, y) \langle f(y), f(x) \rangle$$

for $x, y \in X$. Therefore, there exists an isometry $V : \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{E}$ which satisfies

$$V k_2(\cdot, x) = k_1(\cdot, x) \otimes f(x)$$

for all $x \in X$. A straightforward computation then shows that for $\varphi \in \text{Mult}(\mathcal{H}_1)$ we have

$$(M_\varphi^{\mathcal{H}_1} \otimes I_{\mathcal{E}})^* V k_2(\cdot, x) = V \overline{\varphi(x)} k_2(\cdot, x)$$

for all $x \in X$. In particular, the range of V is coinvariant for $M_\varphi^{\mathcal{H}_1} \otimes I_{\mathcal{E}}$, and the bounded operator $V^*(M_\varphi^{\mathcal{H}_1} \otimes I_{\mathcal{E}})^* V$ on \mathcal{H}_2 has $k_2(\cdot, x)$ as an eigenvector with eigenvalue $\overline{\varphi(x)}$. Thus, we conclude that $\varphi \in \text{Mult}(\mathcal{H}_2)$ and that $V^*(M_\varphi^{\mathcal{H}_1} \otimes I_{\mathcal{E}}) = M_\varphi^{\mathcal{H}_2} V^*$. \square

In the presence of the complete Nevanlinna-Pick property, the last result fits in a chain of equivalences. In particular, we obtain a positive answer to Question 1 for a special class of representations of multiplier algebras of complete Nevanlinna-Pick spaces.

Theorem 4.2. *Let \mathcal{H}_1 and \mathcal{H}_2 be two reproducing kernel Hilbert spaces on the same set X with reproducing kernels k_1, k_2 , respectively. Suppose that \mathcal{H}_1 is an irreducible complete Nevanlinna-Pick space. Then the following assertions are equivalent.*

- (i) *The inclusion $\text{Mult}(\mathcal{H}_1) \hookrightarrow \text{Mult}(\mathcal{H}_2)$ is completely contractive.*
- (ii) *The inclusion $\text{Mult}(\mathcal{H}_1) \hookrightarrow \text{Mult}(\mathcal{H}_2)$ admits a $B(\mathcal{H}_1)$ -coextension.*

- (iii) $\text{Mult}(\mathcal{H}_1) \subset \text{Mult}(\mathcal{H}_2)$ and there exists a Hilbert space \mathcal{E} such that $M_\varphi^{\mathcal{H}_2}$ coextends to $M_\varphi^{\mathcal{H}_1} \otimes I_{\mathcal{E}}$ for every $\varphi \in \text{Mult}(\mathcal{H}_1)$.
- (iv) k_2/k_1 is positive semi-definite on X .

Proof. We first show that (iv) implies (iii) and (ii). By Theorem 4.1, there exists an isometry $V : \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{E}$ such that for every $\varphi \in \text{Mult}(\mathcal{H}_1)$, we have

$$V^*(M_\varphi^{\mathcal{H}_1} \otimes I_{\mathcal{E}}) = M_\varphi^{\mathcal{H}_2} V^*$$

and hence $(M_\varphi^{\mathcal{H}_1} \otimes I_{\mathcal{E}})^* V \mathcal{H}_2 \subset V \mathcal{H}_2$. Let

$$\pi : B(\mathcal{H}_1) \rightarrow B(\mathcal{H}_1 \otimes \mathcal{E}), \quad T \mapsto T \otimes I_{\mathcal{E}}.$$

Then π is a $*$ -representation, and if we identify the space \mathcal{H}_2 with $V \mathcal{H}_2 \subset \mathcal{H}_1 \otimes \mathcal{E}$, it becomes coinvariant under $\pi(M_\varphi^{\mathcal{H}_1})$ and

$$(M_\varphi^{\mathcal{H}_2})^* = (M_\varphi^{\mathcal{H}_1} \otimes I_{\mathcal{E}})^*|_{\mathcal{H}_2} = \pi(M_\varphi^{\mathcal{H}_1})^*|_{\mathcal{H}_2}$$

for $\varphi \in \text{Mult}(\mathcal{H}_1)$.

Both (iii) and (ii) trivially imply (i).

Finally, we show that (i) implies (iv). Observe that both the hypothesis and the conclusion are invariant under rescaling of the kernel k_1 , so we may assume that k_1 is normalized at a point (see [4, Section 2.6]). In this setting, the statement follows from the implication (i) \Rightarrow (ii) of [34, Corollary 3.5] and its proof. \square

For a given algebra of functions \mathcal{M} on a set X , there may be many different reproducing kernel Hilbert spaces \mathcal{H} on X whose multiplier algebra is \mathcal{M} . For instance, the algebra $H^\infty(\mathbb{D})$ of bounded analytic functions on \mathbb{D} is the multiplier algebra of the Hardy space, of the Bergman space and indeed of many more spaces on the unit disc. If one of these spaces is an irreducible complete Nevanlinna-Pick space \mathcal{H}_1 , then the implication (i) \Rightarrow (iii) of Theorem 4.2 shows that among all those spaces, \mathcal{H}_1 is a particularly good choice from the point of view of dilation theory. Indeed, for every $\varphi \in \mathcal{M}$ and each of the spaces \mathcal{H} , the operator $M_\varphi^{\mathcal{H}}$ coextends to $M_\varphi^{\mathcal{H}_1} \otimes I_{\mathcal{E}}$. Therefore, given $\varphi \in \mathcal{M}$, the collection of operators T which can be modelled by $M_\varphi^{\mathcal{H}}$, in the sense that T coextends to $M_\varphi^{\mathcal{H}} \otimes I_{\mathcal{E}}$, is maximal if $\mathcal{H} = \mathcal{H}_1$.

We also observe that in the proof of Theorem 4.2, the Nevanlinna-Pick property of \mathcal{H}_1 was only used to show the implication (i) \Rightarrow (iv), while the implications (iv) \Rightarrow (iii), (iv) \Rightarrow (ii), (iii) \Rightarrow (i) and (ii) \Rightarrow (i) hold without it. Moreover, in many cases of interest, the equivalence of (iii) (in the precise form of Theorem 4.1) and (iv) also holds without the Nevanlinna-Pick assumption on \mathcal{H}_1 , see [24, Theorem 2] and [36, Corollary 6.2]. On the other hand, the following example shows that without the Nevanlinna-Pick assumption on \mathcal{H}_1 , the implications (i) \Rightarrow (ii) and (i) \Rightarrow (iii) (and therefore also (i) \Rightarrow (iv)) may fail.

Example 4.3. Let $\mathcal{H}_1 = A^2$ be the Bergman space on the unit disc whose reproducing kernel is

$$k_1(z, w) = \frac{1}{(1 - z\bar{w})^2}.$$

Let $\mathcal{H}_2 = H^2$ be the classical Hardy space on the unit disc. Then, both $\text{Mult}(A^2)$ and $\text{Mult}(H^2)$ coincide with the algebra $H^\infty(\mathbb{D})$. In particular, the inclusion

$$\text{Mult}(A^2) \hookrightarrow \text{Mult}(H^2)$$

is completely contractive. Let $\pi : B(A^2) \rightarrow B(\mathcal{L})$ be a $*$ -homomorphism which dilates the inclusion map (there always exists such a dilation by Theorem 3.1).

We claim that H^2 cannot be coinvariant for $\pi(\text{Mult}(A^2))$, so that the inclusion does not admit a $C^*(\text{Mult}(A^2))$ -coextension. To this end, observe that the Bergman shift $B = M_z^{A^2}$ satisfies

$$B^2(B^*)^2 - 2BB^* + I \geq 0.$$

This relation is preserved by unital $*$ -homomorphisms as well as by compressions to coinvariant subspaces. Let $S = M_z^{H^2}$ denote the Hardy shift. Since $S = P_{H^2} \pi(B)|_{H^2}$, if H^2 were coinvariant for $\pi(\text{Mult}(A^2))$ we would have

$$S^2(S^*)^2 - 2SS^* + I \geq 0,$$

which is easily verified to be false. Incidentally, this argument also shows that S does not coextend to $B \otimes I_{\mathcal{E}}$ for some Hilbert space \mathcal{E} . Finally, we note that

$$k_2(z, w)/k_1(z, w) = 1 - z\bar{w},$$

which is not positive semi-definite on \mathbb{D} , in accordance with Theorem 4.1.

On the other hand, if we interchange the roles of Bergman space and Hardy space and set $\mathcal{H}_1 = H^2$ and $\mathcal{H}_2 = A^2$, then \mathcal{H}_1 is a complete Nevanlinna-Pick space, the quotient

$$k_2(z, w)/k_1(z, w) = \frac{1}{1 - z\bar{w}} = k_1(z, w)$$

is positive semi-definite, and thus the inclusion $\text{Mult}(H^2) \rightarrow \text{Mult}(A^2)$ admits a $C^*(\text{Mult}(H^2))$ -coextension by Theorem 4.2. \square

This example shows that the complete Nevanlinna-Pick property cannot simply be removed from the statement of Theorem 4.2. In fact, the next result shows that the validity of the implication (i) \Rightarrow (iv) characterizes complete Nevanlinna-Pick spaces.

Theorem 4.4. *Let \mathcal{H}_1 be an irreducible reproducing kernel Hilbert space on a set X with kernel k_1 . Then the following assertions are equivalent.*

- (i) \mathcal{H}_1 is a complete Nevanlinna-Pick space.
- (ii) Whenever \mathcal{H}_2 is another reproducing kernel Hilbert space on X with kernel k_2 such that the inclusion $\text{Mult}(\mathcal{H}_1) \hookrightarrow \text{Mult}(\mathcal{H}_2)$ is completely contractive, then the function k_2/k_1 is positive semi-definite on X .

Proof. In view of Theorem 4.2, it remains to show that (ii) implies (i). Suppose thus that (ii) holds. By rescaling k_1 if necessary, we may without loss of generality assume that k_1 is normalized at a point $x_0 \in X$ so that $k_1(\cdot, x_0) = 1$ (see [4, Section 2.6] and note that the rescaling procedure preserves the complete Nevanlinna-Pick property). Define

$$\mathcal{H}_0 = \{f \in \mathcal{H}_1 : f(x_0) = 0\}.$$

Then \mathcal{H}_0 is a closed subspace of \mathcal{H}_1 and, with the Hilbert space structure inherited from \mathcal{H}_1 , it is a reproducing kernel Hilbert space on X . Since $1 = k_1(\cdot, x_0)$, we see that the reproducing kernel of \mathcal{H}_0 is $k_0 = k_1 - 1$. Some care must be taken here however, since the functions in \mathcal{H}_0 share the common zero x_0 , so that a multiplier on \mathcal{H}_0 is not uniquely determined by its associated multiplication operator. Thus, \mathcal{H}_0 does not fit into the framework of Section 2.

To remedy this situation, we define for $0 < \varepsilon \leq 1$ the kernel

$$k_\varepsilon = k_1 - (1 - \varepsilon) = k_0 + \varepsilon$$

on X and we let \mathcal{H}_ε be the reproducing kernel Hilbert space with kernel k_ε . Clearly, the kernel k_ε does not vanish on the diagonal of $X \times X$, so that the functions in \mathcal{H}_ε have no common zeros. We claim that the inclusion $\text{Mult}(\mathcal{H}_1) \hookrightarrow \text{Mult}(\mathcal{H}_\varepsilon)$ is completely contractive. Let $r \in \mathbb{N}$ and let $\Phi \in M_r(\text{Mult}(\mathcal{H}_1))$ with norm at most 1. Then, Φ can be regarded as a contractive $M_r(\mathbb{C})$ -valued multiplier on \mathcal{H}_1 , so that

$$L_1(z, w) = k_1(z, w)(I_{\mathbb{C}^r} - \Phi(z)\Phi(w)^*)$$

is a positive semi-definite $M_r(\mathbb{C})$ -valued kernel. It is clear that the restriction of Φ to the invariant subspace $\mathcal{H}_0^{(r)}$ also has norm at most 1, hence

$$L_0(z, w) = k_0(z, w)(I_{\mathbb{C}^r} - \Phi(z)\Phi(w)^*)$$

is also positive semi-definite. Therefore,

$$k_\varepsilon(z, w)(I_{\mathbb{C}^r} - \Phi(z)\Phi(w)^*) = \varepsilon L_1(z, w) + (1 - \varepsilon)L_0(z, w)$$

is positive semi-definite, so that Φ belongs to the unit ball of $M_r(\text{Mult}(\mathcal{H}_\varepsilon))$, which proves the claim.

Using (ii) with $\mathcal{H}_2 = \mathcal{H}_\varepsilon$ implies that

$$1 - (1 - \varepsilon)\frac{1}{k_1} = \frac{k_\varepsilon}{k_1}$$

is positive semi-definite on X . Taking the limit as $\varepsilon \rightarrow 0$, we conclude that $1 - 1/k_1$ is positive semi-definite on X , so that k_1 is a complete Nevanlinna-Pick kernel by [4, Theorem 7.31]. \square

We close this section by mentioning a different interpretation of the condition (ii) in Theorem 4.2, which comes from Agler's work on hereditary polynomials. Lemma 3.2 shows that we may reformulate the condition as follows: there exists a unital completely positive linear map $\Psi : B(\mathcal{H}_1) \rightarrow B(\mathcal{H}_2)$ such that

$$\Psi(M_\varphi^{\mathcal{H}_1}(M_\varphi^{\mathcal{H}_1})^*) = M_\varphi^{\mathcal{H}_2}(M_\varphi^{\mathcal{H}_2})^*$$

for all $\varphi \in \text{Mult}(\mathcal{H}_1)$. In addition to Agler, this condition was also considered by Arveson [11, Definition 6.1]. In fact, it is possible to prove the implication (iv) \Rightarrow (iii) in Theorem 4.2 in a way similar to [1], but this approach does not yield the information that the $B(\mathcal{H}_1)$ -coextension can be chosen to be unitarily equivalent to a multiple of the identity representation.

5. COEXTENSIONS FOR UNITARILY INVARIANT SPACES

Whereas Section 4 was devoted to completely contractive inclusions of multiplier algebras of reproducing kernel Hilbert spaces, for the rest of the paper we will be interested in general representations of algebras of multipliers. In this section we once more examine Question 1, this time for the algebras $A(\mathcal{H})$ associated to unitarily invariant complete Nevanlinna-Pick spaces on the ball. This class of spaces is more restricted than that considered in Section 4, but the results we obtain about it are much stronger.

We will make use of a device which has its roots in the work of Agler [1]. He showed that if \mathcal{H} is a reproducing kernel Hilbert space of analytic functions on \mathbb{D} with kernel k , then under suitable assumptions, every operator T which satisfies $\sigma(T) \subset \mathbb{D}$ and $1/k(T, T^*) \geq 0$ coextends to a direct sum of copies of $M_z^{\mathcal{H}}$ [1, Theorem 2.3]. The interpretation of $1/k(T, T^*)$ requires some care in the general case, but we note that if k is the kernel of the Hardy space $H^2(\mathbb{D})$ then $1/k(T, T^*) =$

$1 - TT^*$, so the positivity assumption simply means that T is a contraction. These ideas were extended to spaces on higher dimensional domains $\Omega \subset \mathbb{C}^d$ by Agler and McCarthy [3] and by Ambrozie, Engliš and Müller [5]. As alluded to above, one difficulty with this approach is to make sense of the expression $1/k(T, T^*)$ for general kernels k and tuples of commuting operators T . In [5], a good definition is available whenever the Taylor spectrum of T is contained in Ω , or the function $1/k$ is a polynomial. In our setting however, these conditions are generally not met. Fortunately, for unitarily invariant complete Nevanlinna-Pick spaces, there is a straightforward way to interpret the condition $1/k(T, T^*) \geq 0$.

Let \mathcal{H} be a unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d with kernel k (see Subsection 2.2 for details). Then, for $z, w \in \mathbb{B}_d$ we have that

$$k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n, \quad 1 - \frac{1}{k(z, w)} = \sum_{n=1}^{\infty} b_n \langle z, w \rangle^n$$

for some sequences $\{a_n\}_n, \{b_n\}_n$ of non-negative numbers with $a_0 = 1$ and $a_n > 0$ for every $n \in \mathbb{N}$ (see Lemma 2.1). Alternatively, note that for $z, w \in \mathbb{B}_d$ we have

$$1 - \frac{1}{k(z, w)} = \sum_{n=1}^{\infty} b_n \sum_{|\alpha|=n} \binom{n}{\alpha} z^\alpha \bar{w}^\alpha$$

where for a multi-index of non-negative integers $\alpha = (\alpha_1, \dots, \alpha_d)$ we put

$$\binom{n}{\alpha} = \frac{n!}{\alpha_1! \cdots \alpha_d!}.$$

Given a commuting tuple of operators $T = (T_1, \dots, T_d)$ on a Hilbert space, we write $1/k(T, T^*) \geq 0$ to mean that

$$\sum_{n=1}^N b_n \sum_{|\alpha|=n} \binom{n}{\alpha} T^\alpha (T^*)^\alpha \leq I$$

for all $N \in \mathbb{N}$. Equivalently, $1/k(T, T^*) \geq 0$ if and only if

$$I - \sum_{n=1}^{\infty} b_n \sum_{|\alpha|=n} \binom{n}{\alpha} T^\alpha (T^*)^\alpha \geq 0$$

and the series converges in the strong operator topology.

Remark 5.1. The meaning that we assign to the inequality $1/k(T, T^*) \geq 0$ is consistent with that of Ambrozie-Engliš-Müller. This is manifest when $1/k$ is a polynomial. The other case that these authors consider is when $\sigma(T) \subset \mathbb{B}_d$, where $\sigma(T)$ denotes the Taylor spectrum of T (the book [39] is a standard reference on the topic of the Taylor spectrum). They define an operator $S = 1/k(T, T^*)$ by means of the Taylor functional calculus (see the discussion preceding [5, Lemma 3]). We claim that $S \geq 0$ if and only if $1/k(T, T^*) \geq 0$ in our sense.

First observe that the series $\sum_{n=1}^{\infty} b_n \langle z, w \rangle^n$ converges uniformly on $\overline{\mathbb{B}_d} \times \overline{\mathbb{B}_d}$, as $\sum_{n=1}^{\infty} b_n \leq 1$. Continuity of the Taylor functional calculus (see the discussion after [5, Lemma 3]) therefore shows that

$$S = I - \sum_{n=1}^{\infty} b_n \sum_{|\alpha|=n} \binom{n}{\alpha} T^\alpha (T^*)^\alpha,$$

where the series converges in norm. In particular, S is a positive operator if and only if

$$\sum_{n=1}^{\infty} b_n \sum_{|\alpha|=n} \binom{n}{\alpha} T^\alpha (T^*)^\alpha \leq I,$$

which is our definition of $1/k(T, T^*) \geq 0$. \square

We now record a simple but important fact.

Lemma 5.2. *Let \mathcal{H} be a unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d with kernel k and let $M_z = (M_{z_1}, \dots, M_{z_d})$. Then, $1/k(M_z, M_z^*) \geq 0$.*

Proof. Testing on linear combinations of kernel functions, it is routine to check that for each $N \in \mathbb{N}$, the operator inequality

$$I - \sum_{n=1}^N b_n \sum_{|\alpha|=n} \binom{n}{\alpha} M_z^\alpha (M_z^*)^\alpha \geq 0$$

is equivalent to the function

$$k(z, w) \left(1 - \sum_{n=1}^N b_n \sum_{|\alpha|=n} \binom{n}{\alpha} z^\alpha \overline{w}^\alpha \right)$$

being positive semi-definite on $\mathbb{B}_d \times \mathbb{B}_d$. But since $b_n \geq 0$ for all $n \in \mathbb{N}$, this function dominates

$$k(z, w) \left(1 - \sum_{n=1}^{\infty} b_n \sum_{|\alpha|=n} \binom{n}{\alpha} z^\alpha \overline{w}^\alpha \right)$$

in the natural order of kernels, and the latter function is identically equal to 1, and in particular positive semi-definite. \square

The following slight reformulation of the condition $1/k(T, T^*) \geq 0$ will prove to be convenient. For each multi-index $\alpha \in \mathbb{N}^d \setminus \{0\}$, consider the polynomial

$$\psi_{k, \alpha} = b_{|\alpha|}^{1/2} \binom{|\alpha|}{\alpha} z^\alpha$$

and define the infinite operator tuple

$$\psi_k(T) = (\psi_{k, \alpha}(T))_{\alpha \in \mathbb{N}^d \setminus \{0\}}.$$

It is readily verified that $1/k(T, T^*) \geq 0$ if and only if $\psi_k(T)$ is a row contraction. Consequently, we see that if $1/k(T, T^*) \geq 0$ and ρ is a unital completely contractive representation of an operator algebra containing T_1, \dots, T_d , then $1/k(\rho(T), \rho(T)^*) \geq 0$, where $\rho(T) = (\rho(T_1), \dots, \rho(T_d))$.

Some basic spectral information is encoded in the inequality $1/k(T, T^*) \geq 0$, as we now show. As above, we denote by $\sigma(T) \subset \mathbb{C}^d$ the Taylor spectrum of a commuting tuple of operators $T = (T_1, \dots, T_d)$ on a Hilbert space.

Lemma 5.3. *Let \mathcal{H} be a unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d with kernel k . Let $T = (T_1, \dots, T_d)$ be a commuting tuple of operators on some Hilbert space. If $1/k(T, T^*) \geq 0$, then $\sigma(T) \subset \overline{\mathbb{B}_d}$.*

Proof. Let χ be a character of the unital norm closed operator algebra generated by T_1, \dots, T_d . We claim that

$$\chi(T) = (\chi(T_1), \dots, \chi(T_d)) \in \overline{\mathbb{B}_d}.$$

Since χ is necessarily completely contractive, the discussion preceding the lemma shows that

$$1/k(\chi(T), \chi(T)^*) \geq 0.$$

Thus,

$$\sum_{n=1}^{\infty} b_n \|\chi(T)\|^{2n} = \sum_{n=1}^{\infty} b_n \sum_{|\alpha|=n} \binom{n}{\alpha} |\chi(T)^\alpha|^2 \leq 1.$$

If $\|\chi(T)\| > 1$, then the function $(1 - \sum_{n=1}^{\infty} b_n t^n)^{-1}$ is analytic in a disc centred at the origin of radius strictly greater than 1. On the other hand, recall that

$$\sum_{n=0}^{\infty} a_n t^n = \frac{1}{1 - \sum_{n=1}^{\infty} b_n t^n}$$

and the series on the left-hand side is assumed to have radius of convergence 1 (see Subsection 2.2), which is a contradiction. Thus, $\|\chi(T)\| \leq 1$. Finally, we may apply [39, Proposition 25.3] to conclude that the Taylor spectrum $\sigma(T)$ must lie in $\overline{\mathbb{B}_d}$. \square

If \mathcal{H} is the Hardy space, then $A(\mathcal{H})$ coincides with the disc algebra, which provides an alternate very concrete description of the elements of $A(\mathcal{H})$. In general, such a simple description is not available. At the very least, Lemmas 5.2 and 5.3 show that $\sigma(M_z^{\mathcal{H}}) \subset \overline{\mathbb{B}_d}$, and thus by the basic properties of the Taylor functional calculus we see that $A(\mathcal{H})$ contains all functions which are analytic in a neighbourhood of $\overline{\mathbb{B}_d}$.

We now arrive at a central result of the paper.

Theorem 5.4. *Let \mathcal{H} be a unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d with reproducing kernel k . Let $T = (T_1, \dots, T_d)$ be a commuting tuple of operators on a Hilbert space \mathcal{E} . Then, the following assertions are equivalent.*

- (i) *The tuple T satisfies $1/k(T, T^*) \geq 0$.*
- (ii) *There exists a unital completely contractive homomorphism*

$$\rho : A(\mathcal{H}) \rightarrow B(\mathcal{E})$$

such that $\rho(M_z) = T$.

In this case, the map ρ is necessarily unique and admits a $C^(A(\mathcal{H}))$ -coextension.*

Proof. By Lemma 5.2 we have that $1/k(M_z, M_z^*) \geq 0$, so the discussion following that same lemma shows that (ii) implies (i).

Assume henceforth that (i) holds so that $1/k(T, T^*) \geq 0$. If a map ρ with the required properties exists, it must be unique since $A(\mathcal{H})$ is generated by M_{z_1}, \dots, M_{z_d} as a Banach algebra. To establish the existence of ρ , it suffices to show that there exists a unital completely positive linear map

$$\psi : C^*(A(\mathcal{H})) \rightarrow B(\mathcal{E})$$

with $\psi(M_{z_j}) = T_j$ and $\psi(M_{z_j} M_{z_j}^*) = T_j T_j^*$ for each $j = 1, \dots, d$. For then Lemma 3.2 shows that if

$$\pi : C^*(A(\mathcal{H})) \rightarrow B(\mathcal{L})$$

is a Stinespring dilation of ψ , then \mathcal{E} is coinvariant for $\pi(A(\mathcal{H}))$. Therefore, the map $\rho = \psi|_{A(\mathcal{H})}$ has all the required properties. It thus remains to construct the map ψ . We note that $\sigma(T) \subset \overline{\mathbb{B}_d}$ by Lemma 5.3.

We first consider the case where $\sigma(T) \subset \mathbb{B}_d$. In this setting, [5, Theorems 5 and 6] combined with Remark 5.1 imply that there exists an isometry

$$V : \mathcal{E} \rightarrow \mathcal{H} \otimes \mathcal{E}$$

such that $VT_j^* = (M_{z_j}^* \otimes I_{\mathcal{E}})V$ for all $j = 1, \dots, d$. As in the proof of the implication (iv) \Rightarrow (iii) of Corollary 4.2, it follows that there is a unital $*$ -representation π of $C^*(A(\mathcal{H}))$ such that \mathcal{E} is coinvariant for $\pi(A(\mathcal{H}))$ and

$$P_{\mathcal{E}}\pi(M_{z_j})|_{\mathcal{E}} = T_j \quad (j = 1, \dots, d).$$

Thus, the compression of π to \mathcal{E} yields the desired map ψ by virtue of Lemma 3.2.

Finally, we treat the general case where $\sigma(T) \subset \overline{\mathbb{B}_d}$. For each $0 < r < 1$ we consider the operator tuple $rT = (rT_1, \dots, rT_d)$. We have $\sigma(rT) \subset \mathbb{B}_d$ and it is easy to verify that the assumption $1/k(T, T^*) \geq 0$ implies that $1/k(rT, rT^*) \geq 0$ since $b_n \geq 0$ for all $n \geq 1$. By the previous paragraph, for each $0 < r < 1$ we obtain a unital completely positive map

$$\psi_r : C^*(A(\mathcal{H})) \rightarrow B(\mathcal{E})$$

such that $\psi_r(M_{z_j}) = rT_j$ and $\psi_r(M_{z_j}M_{z_j}^*) = r^2T_jT_j^*$ for each j . Taking a cluster point of the net $\{\psi_r\}_r$ in an appropriate weak topology (the so-called BW-topology, see [40, Lemma 7.2] for instance) yields the required map ψ . \square

As an immediate consequence, we can provide a positive answer to Question 1 for all representations of the algebra $A(\mathcal{H})$ associated to a unitarily invariant complete Nevanlinna-Pick space on the ball.

Corollary 5.5. *Let \mathcal{H} be a unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d . Then every unital completely contractive representation of $A(\mathcal{H})$ admits a $C^*(A(\mathcal{H}))$ -coextension.* \square

Example 4.3 shows that this corollary may fail without the complete Nevanlinna-Pick property, as the Bergman space A^2 is a unitarily invariant space on the disc. Specifically, the implication (ii) \Rightarrow (i) of Theorem 5.4 fails in this case, since $1/k_{A^2}(T, T^*) = 1 - 2TT^* + T^2(T^*)^2$ was seen not to be positive if T is the Hardy shift. We will see in Section 9 that at least for regular spaces, the last corollary in fact characterizes the complete Nevanlinna-Pick property.

We finish this section by deducing an operator theoretic dilation theorem from Theorem 5.4. The first part of this dilation theorem is very much in the spirit of the work of Agler [1], Agler and McCarthy [3] and Ambrozie, Engliš and Müller [5]. The chief difference with their results is that we do not assume that the Taylor spectrum of our operator tuple is contained in the open ball \mathbb{B}_d or that our tuple satisfies a suitable purity condition. Moreover, the Nevanlinna-Pick property once again guarantees that the existence of dilations and of coextensions is equivalent. If $\mathcal{H} = H_d^2$, then the next result is the theorem of Müller and Vasilescu [38] and Arveson [11], which was mentioned in the introduction. Recall from Subsection 2.6 that a unitarily invariant space with kernel $\sum_{n=0}^{\infty} a_n \langle z, w \rangle^n$ is regular if $\lim_{n \rightarrow \infty} a_n / a_{n+1} = 1$.

Theorem 5.6. *Let \mathcal{H} be a regular unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d with reproducing kernel k and let T be a commuting tuple of operators on a Hilbert space \mathcal{E} . Then the following assertions are equivalent.*

- (i) *The tuple T satisfies $1/k(T, T^*) \geq 0$.*
- (ii) *The tuple T coextends to $(M_z^{\mathcal{H}})^{(\kappa)} \oplus U$ for some cardinal κ and a spherical unitary U .*
- (iii) *The tuple T dilates to $(M_z^{\mathcal{H}})^{(\kappa)} \oplus U$ for some cardinal κ and a spherical unitary U .*

Proof. If (i) holds, then Theorem 5.4 yields a $*$ -representation $\pi : C^*(A(\mathcal{H})) \rightarrow B(\mathcal{L})$ such that \mathcal{E} is co-invariant for $\pi(A(\mathcal{H}))$ and $P_{\mathcal{K}}\pi(M_z^{\mathcal{H}})|_{\mathcal{K}} = T$. By Lemma 3.3 and Theorem 2.2, the representation π splits as $\pi = \pi_1 \oplus \pi_2$, where $\pi_1(M_z^{\mathcal{H}})$ is unitarily equivalent to $(M_z^{\mathcal{H}})^{(\kappa)}$ for some cardinal κ and $U = \pi_2(M_z^{\mathcal{H}})$ is a spherical unitary. Thus (ii) follows.

It is trivial that (ii) implies (iii).

Finally, assume that (iii) holds. Observe that the spherical unitary U gives rise to a $*$ -representation of $C(\partial\mathbb{B}_d)$ which maps z to U , hence Theorem 2.2 implies that there exists a $*$ -representation π of $C^*(A(\mathcal{H}))$ which maps $M_z^{\mathcal{H}}$ to $(M_z^{\mathcal{H}})^{(\kappa)} \oplus U$. The compression of π to \mathcal{E} is then a unital completely contractive representation of $A(\mathcal{H})$ which maps $M_z^{\mathcal{H}}$ to T , so that $1/k(T, T^*) \geq 0$ by Theorem 5.4. \square

6. BOUNDARY REPRESENTATIONS AND HYPERRIGIDITY FOR UNITARILY INVARIANT SPACES

In this section, we investigate the notions of boundary representations, C^* -envelope and hyperrigidity (see Subsection 3.5) for the algebras $A(\mathcal{H})$. In particular, we answer Question 2 for these algebras whenever \mathcal{H} is regular.

It is not hard to see that every spherical unitary tuple gives rise to a representation of $A(\mathcal{H})$. We show below via an application of Theorem 3.4 that such a representation has the unique extension property provided that the reproducing kernel is unbounded.

Proposition 6.1. *Let \mathcal{H} be a unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d with reproducing kernel*

$$k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n$$

and let $U = (U_1, \dots, U_d)$ be a spherical unitary on a Hilbert space \mathcal{E} . Then, there exists a unique unital completely contractive homomorphism $\rho : A(\mathcal{H}) \rightarrow B(\mathcal{E})$ such that $\rho(M_z) = U$. If $\sum_{n=0}^{\infty} a_n = \infty$, then ρ has the unique extension property.

Proof. Uniqueness of ρ follows from the fact that $A(\mathcal{H})$ is generated by M_{z_1}, \dots, M_{z_d} as a Banach algebra. To prove existence, recall that the complete Nevanlinna-Pick property implies that there exists a sequence $\{b_n\}_n$ of non-negative numbers such that

$$1 - \frac{1}{k(z, w)} = \sum_{n=1}^{\infty} b_n \langle z, w \rangle^n \quad (z, w \in \mathbb{B}_d).$$

Note that $\sum_{n=1}^{\infty} b_n \leq 1$, and $\sum_{n=1}^{\infty} b_n = 1$ if and only if $\sum_{n=0}^{\infty} a_n = \infty$. Furthermore, recall from the discussion following Lemma 5.2 that for each multi-index

$\alpha \in \mathbb{N}^d \setminus \{0\}$, we defined

$$\psi_{k,\alpha}(z) = b_{|\alpha|}^{1/2} \binom{|\alpha|}{\alpha} z^\alpha.$$

By the Putnam-Fuglede theorem, each $\psi_{k,\alpha}(U)$ is normal and

$$\sum_{\alpha \in \mathbb{N}^d \setminus \{0\}} \psi_{k,\alpha}(U) \psi_{k,\alpha}(U)^* = \sum_{\alpha \in \mathbb{N}^d \setminus \{0\}} b_{|\alpha|} \binom{|\alpha|}{\alpha} U^\alpha (U^*)^\alpha = \sum_{n=1}^{\infty} b_n I.$$

Thus, the tuple $\psi_k(U) = (\psi_{k,\alpha}(U))_\alpha$ is a row contraction and hence $1/k(U, U^*) \geq 0$, so that an application of Theorem 5.4 yields the existence of ρ .

If $\sum_{n=1}^{\infty} b_n = 1$, then $\psi_k(U)$ is a spherical unitary. On the other hand, by Lemma 5.2 we know that $\psi_k(M_z)$ is a row contraction. Since $A(\mathcal{H})$ coincides with the unital norm closed algebra generated by the operators $\psi_{k,\alpha}(M_z)$ for $\alpha \in \mathbb{N}^d \setminus \{0\}$, the representation ρ has the unique extension property by Theorem 3.4. \square

We remark that the mere existence of ρ in the preceding result also follows from the elementary fact that the norm on $A(\mathcal{H})$ dominates the supremum norm over the unit sphere, so that the restriction map $A(\mathcal{H}) \rightarrow C(\partial\mathbb{B}_d)$ is completely contractive.

Moreover, our proof gives a seemingly more general result: if $T = (T_1, \dots, T_d)$ is a commuting tuple of normal operators on a Hilbert space \mathcal{E} such that

$$\sum_{n=1}^{\infty} b_n \sum_{|\alpha|=n} \binom{n}{\alpha} T^\alpha (T^*)^\alpha = I,$$

then there exists a unique unital completely contractive homomorphism

$$\rho : A(\mathcal{H}) \rightarrow B(\mathcal{E})$$

such that $\rho(M_{z_j}) = T_j$ for $1 \leq j \leq d$, and ρ has the unique extension property. However, by applying characters of the C^* -algebra generated by T_1, \dots, T_d to the above equality and invoking Lemma 5.3, one easily sees that these assumptions can only be satisfied if $\sum_{n=0}^{\infty} a_n = \infty$ and T is spherical unitary.

We are now in the position to identify all the boundary representations for $A(\mathcal{H})$ when \mathcal{H} is regular. Observe that Theorem 2.2 shows in particular that if \mathcal{H} is a regular unitarily invariant space, then every $\zeta \in \partial\mathbb{B}_d$ gives rise to a character ρ_ζ on $C^*(A(\mathcal{H}))$ of evaluation at ζ which annihilates the compact operators.

Theorem 6.2. *Let \mathcal{H} be a regular unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d with reproducing kernel*

$$k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n.$$

Suppose that \mathcal{H} is not the Hardy space $H^2(\mathbb{D})$.

- (a) *If $\sum_{n=0}^{\infty} a_n = \infty$, then the boundary representations for $A(\mathcal{H})$ are precisely the characters ρ_ζ for $\zeta \in \partial\mathbb{B}_d$ and the identity representation of $C^*(A(\mathcal{H}))$.*
- (b) *If $\sum_{n=0}^{\infty} a_n < \infty$, then the identity representation of $C^*(A(\mathcal{H}))$ is the only boundary representation for $A(\mathcal{H})$.*

Proof. By Theorem 2.2, we have that $\mathcal{K}(\mathcal{H}) \subset C^*(A(\mathcal{H}))$ and $C^*(A(\mathcal{H}))/\mathcal{K}(\mathcal{H})$ is $*$ -isomorphic to $C(\partial\mathbb{B}_d)$. Lemma 3.3 shows that the only possible irreducible $*$ -representations of $C^*(A(\mathcal{H}))$ are therefore the identity representation and the point evaluation characters ρ_ζ for $\zeta \in \partial\mathbb{B}_d$.

We first show that the identity representation is always a boundary representation for $A(\mathcal{H})$. By Arveson's boundary theorem [9, Theorem 2.11], it suffices to show that the quotient map

$$q : C^*(A(\mathcal{H})) \rightarrow C^*(A(\mathcal{H}))/\mathcal{K}(\mathcal{H}) \cong C(\partial\mathbb{B}^d)$$

is not completely isometric on $A(\mathcal{H})$. We will show that it is not even isometric. First note that $q(M_f) = f$ whence $\|q(M_f)\| = \|f\|_\infty$ for every polynomial f . Next, it follows from [34, Proposition 6.4] that

$$\|M_{z_1^n}\|^2 = \|z_1^n\|_{\mathcal{H}}^2 = 1/a_n,$$

whereas $\|z_1^n\|_\infty = 1$. Thus, q is not isometric unless $a_n = 1$ for all $n \geq 0$. But this would imply \mathcal{H} is the Drury-Arveson space H_d^2 , and it is well known then that q is not isometric if $d \geq 2$ (see [11, Section 3.8]). Indeed,

$$\|M_{z_1 z_2}^{H_d^2}\|^2 = \|z_1 z_2\|_{H_d^2}^2 = 1/2,$$

whereas $\|z_1 z_2\|_\infty^2 = 1/4$. Thus, in order for q to be isometric we must have $d = 1$ and $\mathcal{H} = H_1^2 = H^2(\mathbb{D})$ contrary to our assumption. We conclude that the identity representation of $C^*(A(\mathcal{H}))$ is a boundary representation for $A(\mathcal{H})$.

If $\sum_{n=0}^\infty a_n = \infty$, then Proposition 6.1 shows that the character ρ_ζ of $C^*(A(\mathcal{H}))$ is a boundary representation for $A(\mathcal{H})$ for every $\zeta \in \partial\mathbb{B}_d$.

Finally, assume that $\sum_{n=0}^\infty a_n < \infty$. We must show that no point evaluation character ρ_ζ of $C^*(A(\mathcal{H}))$ is a boundary representation for $A(\mathcal{H})$. In this case, \mathcal{H} can be identified with a reproducing kernel Hilbert space on $\overline{\mathbb{B}_d}$ in a natural way (see Subsection 2.3). Let $\zeta \in \partial\mathbb{B}_d$. Then the restriction $\rho_\zeta|_{A(\mathcal{H})}$ is a unital completely contractive functional which admits two extensions to a state on $C^*(A(\mathcal{H}))$, namely ρ_ζ itself and the state φ defined by

$$\varphi(a) = \frac{\langle ak_\zeta, k_\zeta \rangle}{\|k_\zeta\|^2}, \quad a \in C^*(A(\mathcal{H}))$$

where $k_\zeta = k(\cdot, \zeta) \in \mathcal{H}$. Moreover, these two states are distinct as ρ_ζ annihilates the compacts whereas φ clearly does not. We conclude that ρ_ζ is not a boundary representation for $A(\mathcal{H})$. \square

We remark that the question of whether the identity representation is a boundary representation in the context of unitarily invariant spaces was already considered by Guo, Hu and Xu [31], and a proof of that fact in the setting of Theorem 6.2 could also be based on [31, Proposition 4.9].

Our methods also allow us to determine when the algebras $A(\mathcal{H})$ are hyperrigid. Recall that a unital operator algebra \mathcal{A} is said to be hyperrigid inside of $C^*(\mathcal{A})$ if the restriction of every unital $*$ -representation of $C^*(\mathcal{A})$ to \mathcal{A} has to the unique extension property.

Theorem 6.3. *Let \mathcal{H} be a regular unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d with reproducing kernel*

$$k(z, w) = \sum_{n=0}^\infty a_n \langle z, w \rangle^n.$$

Suppose that \mathcal{H} is not the Hardy space $H^2(\mathbb{D})$.

- (a) *If $\sum_{n=0}^\infty a_n = \infty$, then $A(\mathcal{H})$ is hyperrigid inside of $C^*(A(\mathcal{H}))$.*
- (b) *If $\sum_{n=0}^\infty a_n < \infty$, then $A(\mathcal{H})$ is not hyperrigid inside of $C^*(A(\mathcal{H}))$.*

Proof. (a) By Theorem 2.2 and Lemma 3.3, every unital $*$ -representation π of $C^*(A(\mathcal{H}))$ decomposes as a direct sum $\pi_1 \oplus \pi_2$, where π_1 is unitarily equivalent to a multiple of the identity representation and π_2 annihilates the ideal of compact operators. Since direct sums of representations with the unique extension property have the unique extension property themselves [15, Proposition 4.4], it suffices to show that the restriction to $A(\mathcal{H})$ of the identity representation and of every representation which annihilates the compacts has the unique extension property. The assertion about the identity representation is contained in part (a) of Theorem 6.2. Finally, if π is a representation of $C^*(A(\mathcal{H}))$ which annihilates the compacts, then an application of Theorem 2.2 shows that $(\pi(M_{z_1}), \dots, \pi(M_{z_d}))$ is a spherical unitary, so that $\pi|_{A(\mathcal{H})}$ has the unique extension property by Proposition 6.1.

(b) By part (b) of Theorem 6.2, the restrictions $\rho_\zeta|_{A(\mathcal{H})}$ of the point evaluation characters of $C^*(A(\mathcal{H}))$ do not have the unique extension property. \square

Arveson conjectured that a separable unital operator algebra \mathcal{A} is hyperrigid if and only if every irreducible representation of $C^*(\mathcal{A})$ is a boundary representation for \mathcal{A} [15, Conjecture 4.3]. Combining Theorem 6.2 and Theorem 6.3, we see that for regular unitarily invariant complete Nevanlinna-Pick spaces on the ball, the algebras $A(\mathcal{H})$ support Arveson's conjecture.

Using our knowledge about the boundary representations of $A(\mathcal{H})$, we can also easily identify the C^* -envelope.

Corollary 6.4. *Let \mathcal{H} be a regular unitarily invariant complete Nevanlinna-Pick space on the unit ball. If \mathcal{H} is the Hardy space $H^2(\mathbb{D})$ then $C_e^*(A(\mathcal{H})) = C(\mathbb{T})$, while if it is not, then $C_e^*(A(\mathcal{H})) = C^*(A(\mathcal{H}))$.*

Proof. The case where $\mathcal{H} = H^2(\mathbb{D})$ is classical: $A(H^2(\mathbb{D}))$ is the disc algebra whose Shilov boundary is the unit circle \mathbb{T} . The alternative is a straightforward consequence of the fact that the identity representation is a boundary representation by Theorem 6.2 (see the discussion in Subsection 3.5). \square

We finish this section by observing that the C^* -algebras $C^*(A(\mathcal{H}))$ are in fact isomorphic to one another for every regular unitarily invariant space on the ball. For some classical spaces, this fact is mentioned in [11, Section 5]. We provide a proof below for the reader's convenience.

Proposition 6.5. *Let \mathcal{H} be a regular unitarily invariant space on \mathbb{B}_d . Then, $C^*(A(\mathcal{H}))$ and $C^*(A(H_d^2))$ are unitarily equivalent. More precisely, there exists a unitary $U : H_d^2 \rightarrow \mathcal{H}$ such that the operator*

$$U^* M_{z_j}^{\mathcal{H}} U - M_{z_j}^{H_d^2} \quad (j = 1, \dots, d)$$

is compact, and $U^ C^*(A(\mathcal{H})) U = C^*(A(H_d^2))$.*

Proof. There exists a unique unitary operator

$$U : H_d^2 \rightarrow \mathcal{H}$$

which satisfies

$$Up = \sqrt{a_n} p$$

for every homogeneous polynomial p of degree n and every $n \geq 0$. A simple computation shows that if p is a homogeneous polynomial of degree n , then

$$(U^* M_{z_j}^{\mathcal{H}} U - M_{z_j}^{H_d^2})p = (\sqrt{a_n/a_{n+1}} - 1)z_j p$$

for $j = 1, \dots, d$. By regularity of \mathcal{H} , we know that

$$\lim_{n \rightarrow \infty} a_n / a_{n+1} = 1$$

whence we infer that $U^* M_{z_j}^{\mathcal{H}} U - M_{z_j}^{H_d^2}$ can be approximated in norm by finite rank operators, and thus is compact for every $j = 1, \dots, d$. Hence, the map $T \mapsto U^* T U$ sends $A(\mathcal{H})$ into $A(H_d^2) + \mathcal{K}(H_d^2) \subset C^*(A(H_d^2))$. Clearly, this implies that it sends $C^*(A(\mathcal{H}))$ into $C^*(A(H_d^2))$. By a similar argument, the map $X \mapsto U X U^*$ sends $C^*(A(H_d^2))$ into $C^*(A(\mathcal{H}))$, and the proof is complete. \square

By Corollary 6.4, we conclude that there are only two possible isomorphism classes of C^* -envelopes for the algebras $A(\mathcal{H})$ in this setting.

7. HOMOGENEOUS IDEALS AND ARVESON'S CONJECTURE

In Section 8, we will extend the results of Sections 5 and 6 to quotients of the algebras $A(\mathcal{H})$ by homogeneous ideals. For the moment, we collect here the necessary preliminaries.

Let $\mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d]$ be a proper homogeneous ideal and let \mathcal{H} be a unitarily invariant space on \mathbb{B}_d such that the polynomials are multipliers of \mathcal{H} . We define $\mathcal{H}_{\mathcal{I}} = \mathcal{H} \ominus \mathcal{I}$, which is coinvariant for $A(\mathcal{H})$ (and indeed for $\text{Mult}(\mathcal{H})$), and we put

$$S_j^{\mathcal{I}} = P_{\mathcal{H}_{\mathcal{I}}} M_{z_j}|_{\mathcal{H}_{\mathcal{I}}} \quad (1 \leq j \leq d).$$

Then $S^{\mathcal{I}} = (S_1^{\mathcal{I}}, \dots, S_d^{\mathcal{I}})$ is a commuting tuple of operators on $\mathcal{H}_{\mathcal{I}}$ and it is immediate that $p(S^{\mathcal{I}}) = 0$ for every $p \in \mathcal{I}$. When the ideal \mathcal{I} is apparent from context, we may suppress it from the notation and write simply S instead of $S^{\mathcal{I}}$. We now define $A(\mathcal{H}_{\mathcal{I}})$ as the norm closed algebra generated by $S_1^{\mathcal{I}}, \dots, S_d^{\mathcal{I}}$. The previously studied setting of unitarily invariant spaces on \mathbb{B}_d corresponds to $\mathcal{I} = \{0\}$. We will see in Corollary 8.3 that $A(\mathcal{H}_{\mathcal{I}})$ may alternatively be described as the quotient of $A(\mathcal{H})$ by the norm closure of \mathcal{I} . Part of our motivation for studying these algebras is that they play a role in the recent study of operator theory on varieties [22, 23, 34] and, as we will see, they are connected to Arveson's essential normality conjecture.

Let

$$V(\mathcal{I}) = \{\lambda \in \mathbb{C}^d : p(\lambda) = 0 \text{ for all } p \in \mathcal{I}\}$$

denote the vanishing locus of \mathcal{I} . If the ideal \mathcal{I} is radical, then the space $\mathcal{H}_{\mathcal{I}}$ can be regarded as a reproducing kernel Hilbert space on $V(\mathcal{I}) \cap \mathbb{B}_d$. Indeed, using Hilbert's Nullstellensatz, one can show that the restriction map induces a unitary operator between $\mathcal{H}_{\mathcal{I}}$ and $\mathcal{H}|_{V(\mathcal{I}) \cap \mathbb{B}_d}$ (see [34, Lemma 7.4]). Correspondingly, the algebra $A(\mathcal{H}_{\mathcal{I}})$ is identified with the norm closure of the polynomials in $\text{Mult}(\mathcal{H}|_{V(\mathcal{I}) \cap \mathbb{B}_d})$. In particular, $A(\mathcal{H}_{\mathcal{I}})$ is an algebra of multipliers, so it fits within the scope of the paper. On the other hand, if \mathcal{I} is not a radical ideal, then $A(\mathcal{H}_{\mathcal{I}})$ contains non-zero nilpotent elements, so it is not semi-simple and in particular not an algebra of functions. Nevertheless, we feel that it is worthwhile to include this more general case, since it does not present any additional difficulties and has been studied as well (see, for example, [13, 22]).

While the early results in Section 5 generalize in a straightforward way, Theorem 5.6 and most of the results in Section 6 depend on Theorem 2.2. Generalizing this theorem is very difficult. Indeed, one of its assertions is that the tuple M_z on \mathcal{H} is *essentially normal*: each operator M_{z_j} is normal modulo the ideal of compact operators $\mathcal{K}(\mathcal{H})$. In the case of the Drury-Arveson space, the question of whether

$S^{\mathcal{I}}$ is essentially normal is Arveson's famous essential normality conjecture [12, 16]. Even though this conjecture has witnessed exciting progress recently (see for example [29] and [25]), it remains open in general. We do not address this conjecture directly and instead make the following definition. We say that a homogeneous ideal $\mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d]$ satisfies Arveson's conjecture if the commuting tuple given by

$$P_{H_d^2 \ominus \mathcal{I}} M_{z_j} \Big|_{H_d^2 \ominus \mathcal{I}} \quad (1 \leq j \leq d)$$

is essentially normal.

We will require the following basic property of $C^*(A(\mathcal{H}))$, whose proof is an adaptation of the proof of [13, Proposition 2.5].

Lemma 7.1. *Let \mathcal{H} be a regular unitarily invariant space on \mathbb{B}_d . If $\mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d]$ is a proper homogeneous ideal, then the algebra $A(\mathcal{H}_{\mathcal{I}})$ is irreducible, and indeed, $C^*(A(\mathcal{H}_{\mathcal{I}}))$ contains the ideal of compact operators on $\mathcal{H}_{\mathcal{I}}$.*

Proof. Let $P \in B(\mathcal{H}_{\mathcal{I}})$ be a projection commuting with S_1, \dots, S_d . In particular, if we denote by Q the projection onto the closed subspace $\bigvee_{j=1}^d S_j \mathcal{H}_{\mathcal{I}}$ then $QP = PQ$. Hence $(I - Q)P = P(I - Q)$. Since the polynomials are dense in \mathcal{H} , we see that $(I - Q)\mathcal{H}_{\mathcal{I}}$ is the one dimensional subspace spanned by the constant function $e = 1$, which belongs to $\mathcal{H}_{\mathcal{I}}$ since \mathcal{I} is homogeneous and proper. Thus, P commutes with the rank one projection $e \otimes e$, so that either $Pe = e$ or $Pe = 0$. Given a polynomial f , we have

$$PP_{\mathcal{H}_{\mathcal{I}}}f = Pf(S)e = f(S)Pe.$$

Using once again the polynomials are dense in \mathcal{H} , we see that either $P = I$ or $P = 0$. Thus, $A(\mathcal{H})$ is irreducible.

Finally, the operator $I - \sum_{j=1}^d M_{z_j} M_{z_j}^*$ is compact according to Theorem 2.2. From coinvariance of $\mathcal{H}_{\mathcal{I}}$, we deduce that

$$I_{\mathcal{H}_{\mathcal{I}}} - \sum_{j=1}^d S_j S_j^* = P_{\mathcal{H}_{\mathcal{I}}} \left(I - \sum_{j=1}^d M_{z_j} M_{z_j}^* \right) P_{\mathcal{H}_{\mathcal{I}}}$$

is compact as well. Moreover, this operator is not zero, as

$$\left(I_{\mathcal{H}_{\mathcal{I}}} - \sum_{j=1}^d S_j S_j^* \right) e = e.$$

Thus, $C^*(A(\mathcal{H}_{\mathcal{I}}))$ contains a non-zero compact operator, and hence all compact operators on $\mathcal{H}_{\mathcal{I}}$ since this C^* -algebra is irreducible. \square

We observe that by the Putnam-Fuglede theorem, \mathcal{I} satisfies Arveson's conjecture if and only if $C^*(A(H_d^2 \ominus \mathcal{I}))/\mathcal{K}(H_d^2 \ominus \mathcal{I})$ is a commutative C^* -algebra. Although this property appears at first glance to be specific to the Drury-Arveson space, it is in fact possible to replace H_d^2 with any regular unitarily invariant space. This was shown by Wernet in his PhD thesis [47, Corollary 2.49]. More generally, the following generalization of Proposition 6.5 holds.

Proposition 7.2. *Let \mathcal{H} be a regular unitarily invariant space on \mathbb{B}_d and let $\mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d]$ be a proper homogeneous ideal. Then, the C^* -algebras $C^*(A(\mathcal{H}_{\mathcal{I}}))$ and*

$C^*(A(H_d^2 \ominus \mathcal{I}))$ are unitarily equivalent. In fact, there exists a unitary $U : H_d^2 \ominus \mathcal{I} \rightarrow \mathcal{H}_{\mathcal{I}}$ such that the operator

$$U^* P_{\mathcal{H}_{\mathcal{I}}} M_{z_j}^{\mathcal{H}}|_{\mathcal{H}_{\mathcal{I}}} U - P_{H_d^2 \ominus \mathcal{I}} M_{z_j}^{H_d^2}|_{H_d^2 \ominus \mathcal{I}} \quad (j = 1, \dots, d)$$

is compact, and

$$U^* C^*(A(\mathcal{H}_{\mathcal{I}})) U = C^*(A(H_d^2 \ominus \mathcal{I})).$$

In particular, $P_{\mathcal{H}_{\mathcal{I}}} M_z^{\mathcal{H}}|_{\mathcal{H}_{\mathcal{I}}}$ is essentially normal if and only if $P_{H_d^2 \ominus \mathcal{I}} M_z^{H_d^2}|_{H_d^2 \ominus \mathcal{I}}$ is essentially normal.

Proof. Since \mathcal{I} is homogeneous, the unitary operator $U : H_d^2 \rightarrow \mathcal{H}$ constructed in the proof of Proposition 6.5 maps \mathcal{I} onto \mathcal{I} . Therefore, it maps $H_d^2 \ominus \mathcal{I}$ onto $\mathcal{H}_{\mathcal{I}}$ and

$$U P_{H_d^2 \ominus \mathcal{I}} = P_{\mathcal{H}_{\mathcal{I}}} U,$$

so that for $j = 1, \dots, d$, we obtain

$$U^* P_{\mathcal{H}_{\mathcal{I}}} M_{z_j}^{\mathcal{H}}|_{\mathcal{H}_{\mathcal{I}}} U - P_{H_d^2 \ominus \mathcal{I}} M_{z_j}^{H_d^2}|_{H_d^2 \ominus \mathcal{I}} = P_{H_d^2 \ominus \mathcal{I}} (U^* M_{z_j}^{\mathcal{H}} U - M_{z_j}^{H_d^2})|_{H_d^2 \ominus \mathcal{I}},$$

which is a compact operator by Proposition 6.5. Since both $C^*(A(H_d^2 \ominus \mathcal{I}))$ and $C^*(A(\mathcal{H}_{\mathcal{I}}))$ contain the compact operators by Lemma 7.1, we see as in the proof of Proposition 6.5 that

$$U^* C^*(A(\mathcal{H}_{\mathcal{I}})) U = C^*(A(H_d^2 \ominus \mathcal{I})). \quad \square$$

We remark that Wernet in fact obtained finer results relating the p -essential normality of operator tuples $P_{\mathcal{H}_{\mathcal{I}}} M_z|_{\mathcal{H}_{\mathcal{I}}}$ in different spaces \mathcal{H} , but the version above suffices for our purposes. Whenever Arveson's conjecture holds, we obtain the following generalization of Theorem 2.2.

Theorem 7.3. *Let \mathcal{H} be a regular unitarily invariant space on \mathbb{B}_d and let $\mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d]$ be a proper homogeneous ideal satisfying Arveson's conjecture. Then there exists a short exact sequence*

$$0 \longrightarrow \mathcal{K}(\mathcal{H}_{\mathcal{I}}) \longrightarrow C^*(A(\mathcal{H}_{\mathcal{I}})) \longrightarrow C(V(\mathcal{I}) \cap \partial \mathbb{B}_d) \longrightarrow 0,$$

where the first map is the inclusion and the second map sends S_j to the coordinate function z_j for $1 \leq j \leq d$.

Proof. By [32, Theorem 5.1 (4)] (and the following discussion therein) there is a short exact sequence

$$0 \longrightarrow \mathcal{K}(H_d^2 \ominus \mathcal{I}) \longrightarrow C^*(A(H_d^2 \ominus \mathcal{I})) \longrightarrow C(V(\mathcal{I}) \cap \partial \mathbb{B}_d) \longrightarrow 0.$$

Using this fact along with Proposition 7.2 completes the proof. \square

We use the convention that $C(\emptyset)$ is the zero space. Thus, under the conditions of the previous theorem, if $V(\mathcal{I}) \cap \partial \mathbb{B}_d = \emptyset$ then $C^*(A(\mathcal{H}_{\mathcal{I}})) = \mathcal{K}(\mathcal{H}_{\mathcal{I}})$.

8. ALGEBRAS OF MULTIPLIERS AND HOMOGENEOUS IDEALS

We now turn our efforts to extending the results of Sections 5 and 6 to the algebras $A(\mathcal{H}_I)$ introduced above. While we will seemingly be retreading old ground by proving generalizations of previously stated results, we feel that dealing with the more general setting separately makes for easier reading, in particular given the required preparation (which was taken care of in Section 7).

Our first goal is to generalize Theorem 5.4. To this end, we begin with the following straightforward adaptation of [5, Theorems 5 and 6].

Theorem 8.1. *Let \mathcal{H} be a unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d with reproducing kernel k . Let $\mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d]$ be a proper homogeneous ideal. Let $T = (T_1, \dots, T_d)$ be a commuting tuple of operators on some Hilbert space \mathcal{E} with the following properties:*

- (1) $\sigma(T) \subset \mathbb{B}_d$,
- (2) $1/k(T, T^*) \geq 0$, and
- (3) $p(T) = 0$ for all $p \in \mathcal{I}$.

Then there exists an isometry $V : \mathcal{E} \rightarrow \mathcal{H}_I \otimes \mathcal{E}$ such that

$$(S_j^* \otimes I_{\mathcal{E}})V = VT_j^*$$

for $j = 1, \dots, d$.

Proof. In view of conditions (1) and (2), we may invoke [5, Theorems 5 and 6] to find an isometry $V : \mathcal{E} \rightarrow \mathcal{H} \otimes \mathcal{E}$ such that $(M_{z_j}^* \otimes I_{\mathcal{E}})V = VT_j^*$ for all $j = 1, \dots, d$. Hence

$$(M_p^* \otimes I_{\mathcal{E}})V = Vp(T)^*$$

for every $p \in \mathbb{C}[z_1, \dots, z_d]$. We claim that the range of V is contained in $\mathcal{H}_I \otimes \mathcal{E}$. To this end, let $p \in \mathcal{I}$ and $x, y \in \mathcal{E}$. By (3) we find $p(T) = 0$ and thus

$$\langle Vx, p \otimes y \rangle = \langle (M_p^* \otimes I_{\mathcal{E}})Vx, 1 \otimes y \rangle = \langle Vp(T)^*x, 1 \otimes y \rangle = 0,$$

which establishes the claim. Therefore, we may regard V as an isometry from \mathcal{E} into $\mathcal{H}_I \otimes \mathcal{E}$, and since $\mathcal{H}_I \otimes \mathcal{E}$ is invariant under $M_{z_j}^* \otimes I_{\mathcal{E}}$, we see that

$$(S_j^* \otimes I_{\mathcal{E}})V = VT_j^*$$

for $j = 1, \dots, d$. □

The following result generalizes Theorem 5.4 and Corollary 5.5, and will be one of our central tools.

Theorem 8.2. *Let \mathcal{H} be a unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d with reproducing kernel k . Let $\mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d]$ be a proper homogeneous ideal. Let $T = (T_1, \dots, T_d)$ be a commuting tuple of operators on a Hilbert space \mathcal{E} . Then, the following assertions are equivalent.*

- (i) *The tuple T satisfies $1/k(T, T^*) \geq 0$ and $p(T) = 0$ for every $p \in \mathcal{I}$.*
- (ii) *There exists a unique unital completely contractive homomorphism*

$$\rho : A(\mathcal{H}_I) \rightarrow B(\mathcal{E})$$

such that $\rho(S) = T$.

Moreover, in this case the map ρ admits a $C^(A(\mathcal{H}_I))$ -coextension. In particular, every unital completely contractive representation of $A(\mathcal{H}_I)$ admits a $C^*(A(\mathcal{H}_I))$ -coextension.*

Proof. Assume first that (ii) holds. The composition of the map

$$A(\mathcal{H}) \rightarrow A(\mathcal{H}_{\mathcal{I}}), \quad M_\varphi \mapsto P_{\mathcal{H}_{\mathcal{I}}} M_\varphi|_{\mathcal{H}_{\mathcal{I}}},$$

with ρ is a unital completely contractive representation of $A(\mathcal{H})$ which maps M_z to T , hence $1/k(T, T^*) \geq 0$ by Theorem 5.4. Moreover, $p(T) = \rho(p(S)) = 0$ for every $p \in \mathcal{I}$, and (i) follows.

The proof that (i) implies (ii) and of the fact that ρ admits a $C^*(A(\mathcal{H}_{\mathcal{I}}))$ -coextension is identical to that of the corresponding implication in Theorem 5.4, upon using Theorem 8.1. Note that the fact that \mathcal{I} is homogeneous is needed to ensure that $p(rT) = 0$ for all $p \in \mathcal{I}$ and all $0 < r < 1$. \square

We now make a brief digression to illustrate how the last result can be used to show that $A(\mathcal{H}_{\mathcal{I}})$ is completely isometrically isomorphic to $A(\mathcal{H})/\overline{\mathcal{I}}$, where $\overline{\mathcal{I}} \subset A(\mathcal{H})$ is the closure of \mathcal{I} in the operator norm.

Corollary 8.3. *Let \mathcal{H} be a unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d and let $\mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d]$ be a proper homogeneous ideal. Then the map*

$$\Phi : A(\mathcal{H})/\overline{\mathcal{I}} \rightarrow A(\mathcal{H}_{\mathcal{I}}), \quad M_\varphi + \overline{\mathcal{I}} \mapsto P_{\mathcal{H}_{\mathcal{I}}} M_\varphi|_{\mathcal{H}_{\mathcal{I}}},$$

is a unital completely isometric isomorphism.

Proof. Since \mathcal{I} is contained in the kernel of the unital completely contractive homomorphism

$$A(\mathcal{H}) \rightarrow A(\mathcal{H}_{\mathcal{I}}), \quad M_\varphi \mapsto P_{\mathcal{H}_{\mathcal{I}}} M_\varphi|_{\mathcal{H}_{\mathcal{I}}},$$

the map Φ is a well defined unital completely contractive homomorphism.

By the Blecher-Ruan-Sinclair theorem [40, Corollary 16.7], there exists a unital completely isometric homomorphism $\pi : A(\mathcal{H})/\overline{\mathcal{I}} \rightarrow B(\mathcal{E})$ for some Hilbert space \mathcal{E} . Let $q : A(\mathcal{H}) \rightarrow A(\mathcal{H})/\overline{\mathcal{I}}$ denote the unital completely contractive quotient map, and let $T = \pi(q(M_z))$. Then it is obvious that $p(T) = 0$ for all $p \in \mathcal{I}$, and applying Theorem 5.4 to $\pi \circ q$ shows that $1/k(T, T^*) \geq 0$. Next, Theorem 8.2 yields a unital completely contractive homomorphism $\rho : A(\mathcal{H}_{\mathcal{I}}) \rightarrow B(\mathcal{E})$ which maps $S^{\mathcal{I}}$ to T . Observe that the range of ρ is contained in the range of π , and that $\pi^{-1} \circ \rho$ is a unital completely contractive homomorphism which maps $S^{\mathcal{I}}$ to $M_z + \overline{\mathcal{I}}$. Thus, $\pi^{-1} \circ \rho$ is a completely contractive inverse of Φ , so that Φ is a unital completely isometric isomorphism. \square

Alternatively, a proof of this corollary can be based on the commuting lifting theorem for complete Nevanlinna-Pick spaces of Ball, Trent and Vinnikov [17].

Next, we extract a generalization of Theorem 5.6 from Theorem 8.2. In order to apply Theorem 7.3, we need to require that \mathcal{I} satisfies Arveson's conjecture.

Theorem 8.4. *Let \mathcal{H} be a regular unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d with reproducing kernel k , let $\mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d]$ be a proper homogeneous ideal which satisfies Arveson's conjecture and let T be a commuting tuple of operators on a Hilbert space. Then the following assertions are equivalent.*

- (i) *The tuple T satisfies $1/k(T, T^*) \geq 0$ and $p(T) = 0$ for all $p \in \mathcal{I}$.*
- (ii) *The tuple T coextends to $(S^{\mathcal{I}})^{(\kappa)} \oplus U$ for some cardinal κ and a spherical unitary U whose joint spectrum is contained in $V(\mathcal{I}) \cap \partial \mathbb{B}_d$.*
- (iii) *The tuple T dilates to $(S^{\mathcal{I}})^{(\kappa)} \oplus U$ for some cardinal κ and a spherical unitary U whose joint spectrum is contained in $V(\mathcal{I}) \cap \partial \mathbb{B}_d$.*

Proof. The proof is almost identical to that of Theorem 5.6, we merely need to replace the application of Theorem 5.4 and Theorem 2.2 with an application of Theorem 8.2 and Theorem 7.3, respectively. \square

We remark that for a spherical unitary U , the following assertions are equivalent:

- (i) the joint spectrum of U is contained in $V(\mathcal{I}) \cap \partial\mathbb{B}_d$,
- (ii) $p(U) = 0$ for all $p \in \mathcal{I}$, and
- (iii) $p(U) = 0$ for all $p \in \sqrt{\mathcal{I}}$, the radical of \mathcal{I} .

Thus, Theorem 8.4 could be reformulated using these conditions where appropriate.

Another consequence of Theorem 8.2 is a generalization of Proposition 6.1.

Proposition 8.5. *Let \mathcal{H} be a unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d with reproducing kernel*

$$k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n.$$

Let $\mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d]$ be a proper homogeneous ideal and let $U = (U_1, \dots, U_d)$ be a spherical unitary on some Hilbert space \mathcal{E} whose joint spectrum is contained in $V(\mathcal{I}) \cap \partial\mathbb{B}_d$. Then there exists a unique unital completely contractive homomorphism $\rho : A(\mathcal{H}_{\mathcal{I}}) \rightarrow B(\mathcal{E})$ such that $\rho(S) = U$. If $\sum_{n=0}^{\infty} a_n = \infty$, then ρ has the unique extension property.

Proof. This follows as in the proof of Proposition 6.1 by applying Theorem 8.2 instead of Theorem 5.4. \square

Before we adapt Theorem 6.2, Theorem 6.3 and Corollary 6.4, we need some more preparation. The following fact is standard, but we provide a proof for the sake of completeness.

Lemma 8.6. *Let \mathcal{H} be a unitarily invariant reproducing kernel Hilbert space on \mathbb{B}_d with kernel*

$$k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n$$

and let $\mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d]$ be a homogeneous ideal. If $w \in V(\mathcal{I})$, then $\langle \cdot, w \rangle^n \in \mathcal{H}_{\mathcal{I}}$ and

$$\|\langle \cdot, w \rangle^n\|_{\mathcal{H}_{\mathcal{I}}}^2 = \frac{1}{a_n} \|w\|^{2n}$$

for all $n \in \mathbb{N}$.

Proof. Since \mathcal{H} is unitarily invariant, homogeneous polynomials of different degree are orthogonal, and the homogeneous expansion of a function in \mathcal{H} converges in the norm of \mathcal{H} . In particular, this is true for kernel functions, so if $p \in \mathcal{I}$ is a homogeneous polynomial of degree n , then

$$0 = p(w) = \langle p, k(\cdot, w) \rangle = \sum_{m=0}^{\infty} \langle p, a_m \langle \cdot, w \rangle^m \rangle = a_n \langle p, \langle \cdot, w \rangle^n \rangle.$$

Since \mathcal{I} is homogeneous, it follows that $\langle \cdot, w \rangle^n$ is orthogonal to \mathcal{I} and thus belongs to $\mathcal{H}_{\mathcal{I}}$. If $\|w\| < 1$, then the previous calculation shows that

$$a_n \|\langle \cdot, w \rangle^n\|^2 = \langle \langle \cdot, w \rangle^n, a_n \langle \cdot, w \rangle^n \rangle = \langle \langle \cdot, w \rangle^n, k(\cdot, w) \rangle = \|w\|^{2n},$$

and the general case follows by scaling w . \square

In Theorem 6.2, we had to exclude the case where $\mathcal{H} = H^2(\mathbb{D})$. To handle this exception in the present setting, we require the following well-known observation .

Remark 8.7. If $Y \subset \mathbb{C}^d$ is a subspace of dimension $d' \leq d$ and if $V : \mathbb{C}^{d'} \rightarrow Y$ is a unitary operator, then the composition operator

$$U : H_d^2|_{Y \cap \mathbb{B}_d} \rightarrow H_{d'}^2, \quad f \mapsto f \circ V,$$

is unitary, as

$$\frac{1}{1 - \langle Vz, Vw \rangle_{\mathbb{C}^d}} = \frac{1}{1 - \langle z, w \rangle_{\mathbb{C}^{d'}}}$$

for all $z, w \in \mathbb{B}_{d'}$. Moreover, the map

$$\text{Mult}(H_d^2|_{Y \cap \mathbb{B}_d}) \rightarrow \text{Mult}(H_{d'}^2), \quad M_\varphi \mapsto UM_\varphi U^* = M_{\varphi \circ V},$$

is a completely isometric isomorphism which maps $A(H_d^2|_{Y \cap \mathbb{B}_d})$ onto $A(H_{d'}^2)$. In particular, if V is a one-dimensional subspace, then $H_d^2|_{V \cap \mathbb{B}_d}$ can be identified with the Hardy space $H^2(\mathbb{D})$ on the unit disc in this way. \square

Because of this observation, we say that $\mathcal{H}_{\mathcal{I}}$ is the Hardy space if $\mathcal{H} = H_d^2$ and \mathcal{I} is a radical homogeneous ideal with the property that $V(\mathcal{I})$ is a one-dimensional subspace (recall that if \mathcal{I} is radical, then $\mathcal{H}_{\mathcal{I}}$ can be identified with $\mathcal{H}|_{V(\mathcal{I}) \cap \mathbb{B}_d}$).

We can now generalize Theorem 6.2 and Corollary 6.4. Recall that under the conditions of Theorem 7.3, every $\zeta \in V(\mathcal{I}) \cap \partial \mathbb{B}_d$ gives rise to a character ρ_ζ on $C^*(A(\mathcal{H}_{\mathcal{I}}))$ of evaluation at ζ which annihilates the compact operators.

Theorem 8.8. *Let \mathcal{H} be a regular unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d with reproducing kernel*

$$k(z, w) = \sum_n a_n \langle z, w \rangle^n.$$

Let $\mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d]$ be a proper homogeneous ideal which satisfies Arveson's conjecture. Suppose that $\mathcal{H}_{\mathcal{I}}$ is not the Hardy space.

- (a) *If $\sum_{n=0}^{\infty} a_n = \infty$, then the boundary representations for $A(\mathcal{H}_{\mathcal{I}})$ are precisely the characters ρ_ζ for $\zeta \in V(\mathcal{I}) \cap \partial \mathbb{B}_d$ and the identity representation of $C^*(A(\mathcal{H}_{\mathcal{I}}))$.*
- (b) *If $\sum_{n=0}^{\infty} a_n < \infty$, then the identity representation of $C^*(A(\mathcal{H}_{\mathcal{I}}))$ is the only boundary representation for $A(\mathcal{H}_{\mathcal{I}})$.*

In particular, $C_e^(A(\mathcal{H}_{\mathcal{I}})) = C^*(A(\mathcal{H}_{\mathcal{I}}))$.*

Proof. By Theorem 7.3 and Lemma 3.3, the only possible irreducible representations of $C^*(A(\mathcal{H}_{\mathcal{I}}))$ are the identity representation and the characters corresponding to points in $V(\mathcal{I}) \cap \partial \mathbb{B}_d$ (note that this set may be empty). Proposition 8.5 shows that these characters are boundary representations if $\sum_{n=0}^{\infty} a_n = \infty$. Conversely, suppose that $\sum_{n=0}^{\infty} a_n < \infty$. As remarked in Subsection 2.3, \mathcal{H} can be identified with a reproducing kernel Hilbert space on $\overline{\mathbb{B}_d}$ in this case. Moreover, if $\zeta \in V(\mathcal{I}) \cap \partial \mathbb{B}_d$, then $k_\zeta = k(\cdot, \zeta) \in \mathcal{H}_{\mathcal{I}}$ by the reproducing property of the kernel. We see as in the proof of Theorem 6.2 that the character $\rho_\zeta|_{A(\mathcal{H}_{\mathcal{I}})}$ admits two extensions to a state on $C^*(A(\mathcal{H}_{\mathcal{I}}))$, namely ρ_ζ itself and the vector state associated to $k_\zeta/\|k_\zeta\|$. These clearly differ, so the characters are not boundary representations in this case.

It remains to prove that the identity representation is always a boundary representation, for which we will use Arveson's boundary theorem [9, Theorem 2.11]. By virtue of Theorem 7.3, we have to show that the surjective $*$ -homomorphism

$$\pi : C^*(A(\mathcal{H}_{\mathcal{I}})) \rightarrow C(V(\mathcal{I}) \cap \partial\mathbb{B}_d), \quad S_j \mapsto z_j, \quad (1 \leq j \leq d)$$

is not completely isometric on $A(\mathcal{H}_{\mathcal{I}})$. In fact, we will show that if π is isometric on $A(\mathcal{H}_{\mathcal{I}})$, then $\mathcal{H}_{\mathcal{I}}$ is the Hardy space, contrary to our assumption.

First of all, if π is isometric on $A(\mathcal{H}_{\mathcal{I}})$, then that algebra has no non-zero nilpotent elements, which is easily seen to imply that \mathcal{I} is a radical ideal. If $\mathcal{I} = \langle z_1, \dots, z_d \rangle$, then $\mathcal{H}_{\mathcal{I}}$ is one dimensional so that $C^*(A(\mathcal{H}_{\mathcal{I}}))$ consists of compact operators and the quotient map π is the zero map, which is clearly not isometric on $A(\mathcal{H}_{\mathcal{I}})$. Hence $\mathcal{I} \subsetneq \langle z_1, \dots, z_d \rangle$, so that $V(\mathcal{I}) \cap \partial\mathbb{B}_d$ is not empty by Hilbert's Nullstellensatz. Choose thus $w \in V(\mathcal{I}) \cap \partial\mathbb{B}_d$ and let

$$S_w = \sum_{j=1}^d \overline{w_j} S_j = P_{\mathcal{H}_{\mathcal{I}}} M_{\langle \cdot, w \rangle} \big|_{P_{\mathcal{H}_{\mathcal{I}}}}.$$

It follows from [34, Proposition 6.4] and Lemma 8.6 that $\|S_w^n\|^2 = 1/a_n$ for all $n \in \mathbb{N}$, whereas

$$\max\{|\langle z, w \rangle^n| : z \in V(\mathcal{I}) \cap \partial\mathbb{B}_d\} = 1.$$

Since π is an isometry we infer that $a_n = 1$ for all $n \in \mathbb{N}$, and hence $\mathcal{H} = H_d^2$.

Next, we show that $V(\mathcal{I})$ is a subspace of \mathbb{C}^d . An application of Lemma 8.6 shows that $\langle \cdot, w \rangle \in H_d^2 \ominus \mathcal{I}$ for all $w \in V(\mathcal{I})$, hence for all w in the linear span of $V(\mathcal{I})$. Consequently, for $w \in \text{span}(V(\mathcal{I})) \cap \partial\mathbb{B}_d$, we obtain the lower bound

$$\|S_w\|_{A(H_d^2 \ominus \mathcal{I})} \geq \|S_w 1\|_{H_d^2 \ominus \mathcal{I}} = \|\langle \cdot, w \rangle\|_{H_d^2} = 1.$$

Again, since π is assumed to be isometric, this forces

$$\max\{|\langle z, w \rangle| : z \in V(\mathcal{I}) \cap \partial\mathbb{B}_d\} = 1,$$

which implies by the Cauchy-Schwarz inequality and homogeneity of \mathcal{I} that $w \in V(\mathcal{I})$. Using homogeneity of \mathcal{I} again, we see that the linear span of $V(\mathcal{I})$ is contained in $V(\mathcal{I})$, so that $V(\mathcal{I})$ is a subspace. Since \mathcal{I} is radical, we may use Remark 8.7 to conclude that $\mathcal{H}_{\mathcal{I}}$ is identified with $H_{d'}^2$, and that $A(\mathcal{H}_{\mathcal{I}})$ is completely isometrically isomorphic to $A(H_{d'}^2)$, where $d' = \dim(V(\mathcal{I}))$. If $d' \geq 2$, it follows as in the proof of Theorem 6.2 that π is not isometric, so that $V(\mathcal{I})$ is a one dimensional subspace of \mathbb{C}^d , as asserted. \square

Finally, we obtain a generalization of Theorem 6.3. We say that a homogeneous ideal $\mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d]$ is *relevant* if its radical does not contain the maximal ideal $\langle z_1, \dots, z_d \rangle$ (see [48, Chapter VII]). By Hilbert's Nullstellensatz, \mathcal{I} is relevant if and only if $V(\mathcal{I}) \cap \partial\mathbb{B}_d$ is not empty.

Theorem 8.9. *Let \mathcal{H} be a regular unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d with reproducing kernel*

$$k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n.$$

Let $\mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d]$ be a relevant homogeneous ideal which satisfies Arveson's conjecture. Suppose that $\mathcal{H}_{\mathcal{I}}$ is not the Hardy space.

(a) *If $\sum_{n=0}^{\infty} a_n = \infty$, then $A(\mathcal{H}_{\mathcal{I}})$ is hyperrigid.*

(b) If $\sum_{n=0}^{\infty} a_n < \infty$, then $A(\mathcal{H}_{\mathcal{I}})$ is not hyperrigid.

Proof. (a) The proof of Theorem 6.3 carries over to this setting. Indeed, Theorem 7.3 and Lemma 3.3 show that every representation π of $C^*(A(\mathcal{H}_{\mathcal{I}}))$ splits as a direct sum of $\pi_1 \oplus \pi_2$, where π_1 is unitarily equivalent to a multiple of the identity representation and π_2 annihilates the compact operators. The identity representation is a boundary representation by Theorem 8.8, so it remains to show that $\pi_2|_{A(\mathcal{H}_{\mathcal{I}})}$ has the unique extension property. Using Theorem 7.3 again, we see that $U = (\pi_2(S_1), \dots, \pi_2(S_d))$ is a spherical unitary whose joint spectrum is contained in $V(\mathcal{I}) \cap \partial\mathbb{B}_d$, so the assertion follows from Proposition 8.5.

(b) Since \mathcal{I} is relevant, the set $V(\mathcal{I}) \cap \partial\mathbb{B}_d$ is not empty, hence part (b) of Theorem 8.8 shows that the point evaluation characters $\rho_{\zeta}|_{A(\mathcal{H}_{\mathcal{I}})}$ with $\zeta \in V(\mathcal{I}) \cap \partial\mathbb{B}_d$ do not have the unique extension property. \square

We remark that if $\mathcal{I} \subset \mathbb{C}[z_1, \dots, z_d]$ is a proper homogeneous ideal which is not relevant, then \mathcal{I} contains a power of the maximal ideal $\langle z_1, \dots, z_d \rangle$, hence $\mathcal{H}_{\mathcal{I}}$ is finite dimensional. Lemma 7.1 therefore shows that $C^*(A(\mathcal{H}_{\mathcal{I}})) = \mathcal{K}(\mathcal{H}_{\mathcal{I}}) = B(\mathcal{H}_{\mathcal{I}})$ in this case, so $A(\mathcal{H}_{\mathcal{I}})$ is always hyperrigid (for instance by Arveson's boundary theorem), regardless of the properties of the sequence $\{a_n\}_n$.

As in Section 6, we see that Theorems 8.8 and 8.9 combined show that the algebras $A(\mathcal{H}_{\mathcal{I}})$ support Arveson's hyperrigidity conjecture, provided that \mathcal{I} satisfies Arveson's essential normality conjecture. If $\mathcal{H} = H_d^2$, this is proved in [35] (see Proposition 4.11 and Theorem 4.12 in that paper). While the authors also extend their results to other unitarily invariant spaces [35, Section 5], their setting is rather different from ours and there is thus very little overlap with our present work. For instance, this can be seen by comparing our standard examples of Subsection 2.4 to their examples in [35, Section 5.2].

More precisely, while they do not assume that their spaces have the complete Nevanlinna-Pick property, they require that the tuple $(M_{z_1}, \dots, M_{z_d})$ is a row contraction. It is not hard to see that the Drury-Arveson space is the only unitarily invariant complete Nevanlinna-Pick space on \mathbb{B}_d for which this happens. Indeed, if $\|M_{z_1}\| \leq 1$, then $a_1 \geq 1$ (for instance by Lemma 8.6). On the other hand, if \mathcal{H} is a unitarily invariant complete Nevanlinna-Pick space, then by Lemma 2.1 we have

$$1 - \frac{1}{k(z, w)} = \sum_{n=1}^{\infty} b_n \langle z, w \rangle^n$$

with $b_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n \leq 1$. It is easily verified that $a_1 = b_1$, which forces $b_1 = 1$ and $b_n = 0$ for $n \geq 2$, so that \mathcal{H} is the Drury-Arveson space. Thus our setting and that of [35] are almost complementary.

9. NECESSITY OF THE NEVANLINNA-PICK PROPERTY FOR COEXTENSIONS

We saw in Corollary 5.5 that if \mathcal{H} is a unitarily invariant complete Nevanlinna-Pick space on the unit ball, then every unital completely contractive representation of $A(\mathcal{H})$ admits a $C^*(A(\mathcal{H}))$ -coextension, and we generalized this result to the algebras $A(\mathcal{H}_{\mathcal{I}})$ in Theorem 8.2. We also observed in Example 4.3 that this automatic coextension property fails for the Bergman space on the unit disc. In this section, we will show that under some mild additional assumptions, the automatic coextension property in fact characterizes those unitarily invariant spaces which enjoy the complete Nevanlinna-Pick property.

Throughout this section, \mathcal{H} denotes a unitarily invariant space on \mathbb{B}_d whose multiplier algebra contains the polynomials. Recall then that $A(\mathcal{H})$ is the norm closure of the polynomial multipliers. Let

$$k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n$$

be the reproducing kernel of \mathcal{H} , with $a_0 = 1$ and $a_n > 0$ for all $n \in \mathbb{N}$. Then, there is a unique sequence $\{b_n\}_n$ of real numbers such that

$$(1) \quad 1 - \frac{1}{\sum_{n=0}^{\infty} a_n t^n} = \sum_{n=1}^{\infty} b_n t^n$$

holds for all t in a neighbourhood of 0.

We first examine the case where $1/k$ is a polynomial, so that all but finitely many of the numbers b_n are equal to 0. In this case, we can use the main idea of the proof of Theorem 4.4 to show that the complete Nevanlinna-Pick property is necessary for Corollary 5.5 to hold.

Proposition 9.1. *Let \mathcal{H} be a unitarily invariant space on the unit ball such that the polynomials are multipliers. Let k be the reproducing kernel of \mathcal{H} and assume that $1/k$ is a polynomial. If every unital completely contractive representation of $A(\mathcal{H})$ admits a $C^*(A(\mathcal{H}))$ -coextension, then \mathcal{H} is an irreducible complete Nevanlinna-Pick space.*

Proof. Let \mathcal{H}_0 be the closed subspace of all functions in \mathcal{H} which vanish at 0. Clearly, \mathcal{H}_0 is invariant for all multiplication operators, hence the restriction map

$$A(\mathcal{H}) \rightarrow B(\mathcal{H}_0), \quad M_\varphi \mapsto M_\varphi|_{\mathcal{H}_0},$$

is a unital completely contractive representation of $A(\mathcal{H})$. By assumption, this representation admits a $C^*(A(\mathcal{H}))$ -coextension π .

Given a commuting tuple of operators $T = (T_1, \dots, T_d)$ on a Hilbert space, define

$$1/k(T, T^*) = I - \sum_{n=1}^{\infty} b_n \sum_{|\alpha|=n} \binom{n}{\alpha} T^\alpha (T^*)^\alpha.$$

Observe that since $1/k$ is polynomial, the sum is in fact finite. Moreover, Equation (1) holds for all $t \in \mathbb{D}$, so that arguing as in the proof of Lemma 5.2, we see that $1/k(M_z, M_z^*) \geq 0$, hence $1/k(\pi(M_z), \pi(M_z)^*) \geq 0$. Let $T = M_z|_{\mathcal{H}_0}$. Since T is the compression of $\pi(M_z)$ to the coinvariant subspace \mathcal{H}_0 , it follows that $1/k(T, T^*) \geq 0$.

Let $k_0 = k - 1$ which is the reproducing kernel of \mathcal{H}_0 . Then

$$T_j^* k_0(\cdot, w) = \overline{w_j} k_0(\cdot, w) \quad (j = 1, \dots, d).$$

Applying the inequality $1/k(T, T^*) \geq 0$ to arbitrary finite linear combinations of the kernel functions $k_0(\cdot, w)$, $w \in \mathbb{B}_d$, we see that the function

$$1 - \frac{1}{k} = \frac{k_0}{k}$$

is positive semi-definite on \mathbb{B}_d . It is well known that this implies that $b_n \geq 0$ for each $n \geq 1$ (see for instance [34, Corollary 6.3]) so that \mathcal{H} is an irreducible complete Nevanlinna-Pick space by Lemma 2.1. \square

In the preceding proof, the observation that $1/k(M_z, M_z^*)$ played a crucial role. If $1/k$ is not a polynomial, then it is not as easy to define $1/k(M_z, M_z^*)$. To overcome this difficulty, we instead consider compressions $S^\mathcal{I}$ of M_z to the orthogonal complement of homogeneous ideals \mathcal{I} of finite codimension. These compressions are nilpotent, so that the definition of $1/k(S^\mathcal{I}, (S^\mathcal{I})^*)$ is once again straightforward.

To be precise, suppose that $T = (T_1, \dots, T_d)$ is a commuting tuple of nilpotent operators. Then we define

$$1/k(T, T^*) = I - \sum_{n=1}^{\infty} b_n \sum_{|\alpha|=n} \binom{n}{\alpha} T^\alpha (T^*)^\alpha.$$

Observe that since each T_j is nilpotent, the sum is in fact finite. The next lemma is the main technical tool of this section.

Lemma 9.2. *Let $N \in \mathbb{N}$, let*

$$\mathcal{I} = \langle z_1, \dots, z_d \rangle^{N+1} \subset \mathbb{C}[z_1, \dots, z_d]$$

and let

$$(\mathcal{H}_I)_0 = \{f \in \mathcal{H}_I : f(0) = 0\}.$$

Put $S = S^\mathcal{I}$ and $T = S|_{(\mathcal{H}_I)_0}$. Then \mathcal{H}_I consists of all polynomials of degree at most N . Moreover,

- (a) $1/k(S, S^*) \geq 0$, and
- (b) $1/k(T, T^*)p = \frac{b_m}{a_m}p$ for every homogeneous polynomial p of degree $1 \leq m \leq N$.

Proof. Since homogeneous polynomials of different degree are orthogonal, it follows that \mathcal{H}_I consists of all polynomials of degree at most N . We first show that if $n \geq 1$ and if p is a homogeneous polynomial of degree $m \geq n$, then

$$(2) \quad \sum_{|\alpha|=n} \binom{n}{\alpha} M_z^\alpha (M_z^*)^\alpha p = \frac{a_{m-n}}{a_m} p.$$

To this end, recall that the space of homogeneous polynomials of degree m is spanned by the polynomials $\langle z, w \rangle^m \in \mathbb{C}[z_1, \dots, z_d]$, where $w \in \mathbb{B}_d$. Moreover, using the fact that $M_{z_j}^*$ maps homogeneous polynomials of degree m to homogeneous polynomials of degree $m-1$ and that $M_{z_j}^* k(\cdot, w) = \overline{w_j} k(\cdot, w)$, one readily sees that

$$M_{z_j}^* \langle z, w \rangle^m = \frac{a_{m-1}}{a_m} \overline{w_j} \langle z, w \rangle^{m-1}$$

for each $j = 1, \dots, d$. Consequently, if $|\alpha| = n$, then

$$(M_z^\alpha)^* \langle z, w \rangle^m = \frac{a_{m-n}}{a_m} \overline{w}^\alpha \langle z, w \rangle^{m-n},$$

so that

$$\sum_{|\alpha|=n} \binom{n}{\alpha} M_z^\alpha (M_z^*)^\alpha \langle z, w \rangle^m = \frac{a_{m-n}}{a_m} \sum_{|\alpha|=n} \binom{n}{\alpha} z^\alpha \overline{w}^\alpha \langle z, w \rangle^{m-n} = \frac{a_{m-n}}{a_m} \langle z, w \rangle^m,$$

which proves the claim.

Observe further that multiplying Equation (1) by $\sum_{n=0}^{\infty} a_n t^n$ and comparing coefficients yields that

$$(3) \quad \sum_{n=1}^m b_n a_{m-n} = a_m$$

for $m \geq 1$.

With these two observations in hand, the remainder of the proof is now a straightforward computation. Indeed, to prove (a), recall that $\mathcal{H}_{\mathcal{I}}$ consists of all polynomials of degree at most N . If p is a homogeneous polynomial of degree $m \leq N$, then

$$S^\alpha(S^*)^\alpha p = \begin{cases} M_z^\alpha(M_z^*)^\alpha p & \text{if } m \geq |\alpha|, \\ 0 & \text{if } m < |\alpha|, \end{cases}$$

so that if $1 \leq m \leq N$, then

$$\sum_{n=1}^N b_n \sum_{|\alpha|=n} \binom{n}{\alpha} S^\alpha(S^*)^\alpha p = \sum_{n=1}^m b_n \sum_{|\alpha|=n} \binom{n}{\alpha} M_z^\alpha(M_z^*)^\alpha p = \sum_{n=1}^m b_n \frac{a_{m-n}}{a_m} p = p,$$

by Equations (2) and (3). If the degree m of p is equal to 0, then the left-hand side is equal to 0, so that

$$1/k(S, S^*) = I - \sum_{n=1}^N b_n \sum_{|\alpha|=n} \binom{n}{\alpha} S^\alpha(S^*)^\alpha$$

is the orthogonal projection onto the one dimensional space $\mathbb{C}1$. In particular, $1/k(S, S^*) \geq 0$.

To prove (b), observe that $(\mathcal{H}_{\mathcal{I}})_0$ is spanned by the homogeneous polynomials of degree $1 \leq m \leq N$. Moreover, if p is a homogeneous polynomial of degree $1 \leq m \leq N$, then

$$T^\alpha(T^*)^\alpha p = S^\alpha P_{(\mathcal{H}_{\mathcal{I}})_0}(S^*)^\alpha p = \begin{cases} M_z^\alpha(M_z^*)^\alpha p & \text{if } m \geq |\alpha| + 1, \\ 0 & \text{if } m < |\alpha| + 1. \end{cases}$$

Consequently,

$$\begin{aligned} (I - 1/k(T, T^*))p &= \sum_{n=1}^N b_n \sum_{|\alpha|=n} \binom{n}{\alpha} T^\alpha(T^*)^\alpha p \\ &= \sum_{n=1}^{m-1} b_n \sum_{|\alpha|=n} \binom{n}{\alpha} M_z^\alpha(M_z^*)^\alpha p \\ &= \sum_{n=1}^{m-1} b_n \frac{a_{m-n}}{a_m} p = \frac{a_m - b_m}{a_m} p, \end{aligned}$$

where we have again used Equations (2) and (3). This identity proves (b). \square

With these preliminary observations in hand, we can now show that the last statement of Theorem 8.2 characterizes those unitarily invariant spaces which have the complete Nevanlinna-Pick property.

Proposition 9.3. *Let \mathcal{H} be a unitarily invariant space on the unit ball such that the polynomials are multipliers. Suppose that for every proper homogeneous ideal of polynomials \mathcal{I} , every unital completely contractive representation of $A(\mathcal{H}_{\mathcal{I}})$ admits a $C^*(A(\mathcal{H}_{\mathcal{I}}))$ -coextension. Then \mathcal{H} is an irreducible complete Nevanlinna-Pick space.*

Proof. We apply the assumption to the ideals $\mathcal{I} = \langle z_1, \dots, z_d \rangle^{N+1}$ for $N \in \mathbb{N}$. Let $S = S^{\mathcal{I}}$, let $(\mathcal{H}_{\mathcal{I}})_0$ be the space of all functions in $\mathcal{H}_{\mathcal{I}}$ which vanish at 0 and

let $T = S|_{(\mathcal{H}_{\mathcal{I}})_0}$. Part (a) of Lemma 9.2 shows that $1/k(S, S^*) \geq 0$. Now, by assumption, the unital completely contractive representation

$$A(\mathcal{H}_{\mathcal{I}}) \rightarrow B((\mathcal{H}_{\mathcal{I}})_0), \quad M_{\varphi} \mapsto M_{\varphi}|_{(\mathcal{H}_{\mathcal{I}})_0},$$

admits a $C^*(A(\mathcal{H}_{\mathcal{I}}))$ -coextension. Hence we must also have $1/k(T, T^*) \geq 0$. By part (b) of Lemma 9.2, this forces $b_n \geq 0$ for $1 \leq n \leq N$, since $a_n > 0$. Since $N \in \mathbb{N}$ was arbitrary, we conclude that \mathcal{H} is an irreducible complete Nevanlinna-Pick space by Lemma 2.1. \square

If we assume in addition that \mathcal{H} is regular, then it suffices to consider $A(\mathcal{H})$ itself. Recall that for a regular unitarily invariant space, the polynomials are automatically multipliers.

Theorem 9.4. *Let \mathcal{H} be a regular unitarily invariant space on \mathbb{B}_d . If every unital completely contractive representation of $A(\mathcal{H})$ admits a $C^*(A(\mathcal{H}))$ -coextension, then \mathcal{H} is an irreducible complete Nevanlinna-Pick space.*

Proof. Let $N \in \mathbb{N}$ and let $\mathcal{I} = \langle z_1, \dots, z_d \rangle^{N+1} \subset \mathbb{C}[z_1, \dots, z_d]$. As before, let $S = S^{\mathcal{I}}$, let $(\mathcal{H}_{\mathcal{I}})_0$ be the space of all functions in $\mathcal{H}_{\mathcal{I}}$ which vanish at 0 and let

$$T = P_{(\mathcal{H}_{\mathcal{I}})_0} M_z|_{(\mathcal{H}_{\mathcal{I}})_0}.$$

Since $(\mathcal{H}_{\mathcal{I}})_0$ is semi-invariant for $A(\mathcal{H})$, the map

$$A(\mathcal{H}) \rightarrow B((\mathcal{H}_{\mathcal{I}})_0), \quad M_{\varphi} \mapsto P_{(\mathcal{H}_{\mathcal{I}})_0} M_{\varphi}|_{(\mathcal{H}_{\mathcal{I}})_0},$$

is a unital completely contractive representation, and thus it admits a $C^*(A(\mathcal{H}))$ -coextension π . Since \mathcal{H} is regular, Theorem 2.2 and Lemma 3.3 show that $\pi = \pi_1 \oplus \pi_2$, where π_1 is unitarily equivalent to a multiple of the identity representation and $\pi_2(M_z)$ is a spherical unitary. Consequently, there exist Hilbert spaces \mathcal{E} and \mathcal{L} , a spherical unitary $U = (U_1, \dots, U_d)$ on \mathcal{L} and an isometry

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} : (\mathcal{H}_{\mathcal{I}})_0 \rightarrow (\mathcal{H} \otimes \mathcal{E}) \oplus \mathcal{L}$$

such that

$$V^*[(M_{z_j} \otimes I_{\mathcal{E}}) \oplus U_j] = T_j V^*,$$

and hence

$$V_1^*(M_{z_j} \otimes I_{\mathcal{E}}) = T_j V_1^* \quad \text{and} \quad V_2^* U_j = T_j V_2^*$$

for $j = 1, \dots, d$. Since $T^{\alpha} = 0$ if $|\alpha| = N+1$, we deduce that

$$V_2^* V_2 = V_2^* \sum_{|\alpha|=N+1} \binom{N+1}{\alpha} U^{\alpha} (U^*)^{\alpha} V_2 = \sum_{|\alpha|=N+1} \binom{N+1}{\alpha} T^{\alpha} V_2^* V_2 (T^{\alpha})^* = 0,$$

hence $V_2 = 0$. Furthermore, if $p \in \mathcal{I}$ then $p(T) = 0$, so arguing as in the proof of Theorem 8.1, it follows that the range of V_1 is contained in $\mathcal{H}_{\mathcal{I}} \otimes \mathcal{E}$. Consequently, V maps $(\mathcal{H}_{\mathcal{I}})_0$ into $(\mathcal{H}_{\mathcal{I}}) \otimes \mathcal{E}$, and T coextends to $S^{\mathcal{I}} \otimes I_{\mathcal{E}}$. As in the last paragraph of the proof of Proposition 9.3, an application of Lemma 9.2 now shows that \mathcal{H} is an irreducible complete Nevanlinna-Pick space. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG, MANITOBA, CANADA R3T 2N2

E-mail address: raphael.clouatre@umanitoba.ca

DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY IN ST. LOUIS, ONE BROOKINGS DRIVE, ST. LOUIS, MO 63130, USA

E-mail address: mpartz@wustl.edu